Within the framework of the LIBOR market model, this article presents a new approach for finding the approximate distribution of constant maturity swap (CMS) rates under forward martingale measures. With this approach, many popular CMS-type interest rate derivatives, such as CMS, CMS caps, CMS floors, CMS steepeners, and CMS range accruals, can be priced under the LIBOR market model and their risk can be managed consistently with LIBOR-type interest rate derivatives. We use this approximation to price three types of CMSs, namely, CMS-for-CMS, CMS-for-LIBOR, and CMS-for-fixed CMSs. The resulting pricing formulas are shown to be robustly accurate and time saving by comparison with Monte Carlo simulations based on the market data over a recent three-year period.

A constant maturity swap (CMS) is an agreement that designates two counterparties to periodically exchange two payment streams (commonly called “legs”) over a specified period. Based on an agreed notional principal, one leg pays (receives) a CMS rate, while the other receives (pays) a fixed or floating rate. The floating rate may be a London Interbank Offer Rate (LIBOR) or another swap rate with a different tenor. CMSs started trading in the mid-1990s, and their trading volume has continued to grow rapidly.

There are two main types of CMS end-users. The first type includes investors who wish to take profits from a change in the shape of the yield curve. In a flat yield curve environment, market participants may take a view that long-term CMS rates will rise in the future as the yield curve steepens and thus take profits by entering CMS positions to receive long-term CMS rates and to pay short-term CMS rates. For example, as shown in Exhibit 1, the spreads between the 10-year and 2-year U.S. dollar CMS rates in January 1995, June 2000, and March 2006 were lower than 30 basis points (bps), which is significantly narrower than they used to be. Due to the flatness of the yield curve, investors were able to generate attractive returns as the yield curve steepened. In July 2003, the spread reached its peak and was higher than 260 bps. Thus, CMSs could also be used to take profits in a very steep yield-curve environment.

The second type of end-users includes corporations and investors who seek to flexibly and efficiently maintain a constant asset or liability duration. For example, life insurers may be heavily indebted to long-term payment obligations, encountering risk if the back end of the yield curve rises sharply. To hedge this risk, they may use CMSs to swap their assets from those receiving short-term interest rates (e.g., 6-month LIBOR) to those receiving long-term swap rates (e.g., 10-year swap rate). With all these uses, it is not surprising that CMSs are actively traded in financial markets.
As indicated by Jamshidian [1997], under a forward swap measure, a forward swap rate is a martingale and, thus, the expectation of a CMS rate. But CMSs are usually priced under forward measures rather than under forward swap measures, thereby making their pricing more complicated. Actually, within most interest rate models, the distribution of CMS rates under forward measures is unknown. Therefore, the expectation of cash flows linked to CMSs cannot be computed analytically.

In practice, two main approaches are used to compute the expectation of CMS rates. The first approach is based on Monte Carlo simulation within an interest rate model. While this approach is flexible enough to compute prices of almost every kind of interest rate derivative, its disadvantage is that it is too time consuming, especially in a competitive financial market. The second approach is to compute the expectation of CMS rates via an approximate analytic formula, namely, the forward swap rate (obtainable from market data) multiplied by a convexity adjustment. Many CMS convexity adjustment formulas are available in the literature, such as Benhamou [2000], Pugachevsky [2001], Brigo and Mercurio [2006], and Hull [2006]. Some of them are based on less theoretically sound assumptions and may lead to some pricing errors. Increasing competition in the CMS market, however, has made the inaccuracies of the conventional CMS convexity adjustments more apparent. Accordingly, this article attempts to develop an alternative method to compute the expectation of CMS rates under forward measures.

The LIBOR market model (LMM) and the swap market model (SMM) are well known to be incompatible, and thus the distribution of swap rates within the LMM framework is unknown. We present an alternative approach for finding a lognormal distribution to approximate the distribution of a future CMS rate under a forward measure. With this approach, many popular CMS-type interest rate derivatives, such as CMSs, CMS caps, CMS floors, CMS steepeners, and CMS range accruals, can be priced in the LMM and their risk management can be conducted consistently with LIBOR-type interest rate derivatives. This article intends to price three types of CMS contracts within the LMM, namely, CMS-for-CMS, CMS-for-LIBOR, and CMS-for-fixed-CMS. The resulting pricing formulas are shown to be robustly accurate and time saving by comparison with Monte Carlo simulations based on the market data over the recent three years.

This article is organized as follows. The next section reviews the LMM and introduces some useful techniques, such as the change of measure and the lognormalization of LIBOR rates and swap rates. The third section presents three types of CMSs and their pricing formulas. The fourth section examines the accuracy of the approximate formulas via Monte Carlo simulations, and the last section summarizes the results.

**EXHIBIT 1**
The Spread between the 10-Year and 2-Year CMS Rates

![Graph showing the spread between the 10-year and 2-year CMS rates from 1989 to 2008.]

*Note: The time series data of the spread in basis points between the 10-year and 2-year CMS rates from July 3, 1989, to October 31, 2008.*

**REVIEW OF THE LMM AND AN APPROXIMATE DISTRIBUTION OF A SWAP RATE IN THE LMM**

In the first subsection, we briefly review the LIBOR market model and some useful techniques, such as the change of measure and the lognormalization of LIBOR rates. In the second subsection, we introduce a new lognormalization approach for swap rates under the LMM.

**Review of the LMM**

This subsection briefly reviews the LMM developed by Brace, Gatarek, and Musiela (BGM) [1997], Miltersen,
Sandmann, and Sondermann [1997], and Musiela and Rutkowski [1997]. The LMM has been widely used in the marketplace for several reasons. It directly specifies the behavior of the market-observable rate (LIBOR) rather than the abstract rates in the traditional interest rate models, thus making the model richer in financial intuition. The LIBOR modeled in the LMM follows a lognormal distribution, which avoids negative rates and pricing errors.1 Moreover, the pricing formulas of caps and floors within the LMM framework are Black's formulas, thus making the calibration procedure easier. In addition, most actively traded interest rate products can be priced within the LMM framework so that their hedges can be managed consistently and efficiently.2

The LMM is briefly introduced as follows, and we will employ it to price CMSs in the next section. Assume that trading takes place continuously over an interval [0, T], 0 < T < ∞. The uncertainty is described by the filtered spot martingale probability space (Ω, F, Q, {Ft}t∈[0,T]), and an m-dimensional independent standard Brownian motion W(t) = (W₁(t), W₂(t), ..., Wₘ(t)) is defined on this filtered probability space. The information flow, accruing to all the agents in the economy, is represented by the filtration {Ft}t∈[0,T], which satisfies the usual hypotheses.3 We introduce some notations as follows. Q denotes the spot martingale probability measure. P(t, T) denotes the time-t price of a zero-coupon bond paying $1 at time T. L(t, T) is forward LIBOR contracted at time t for the period [t, T + δ]. Q̂ denotes the forward martingale measure with respect to the numéraire P(t, T). The relationship between L(t, T) and P(t, T) can be expressed by

\[ L(t, T) = (P(t, T) - P(t, T + δ))/δP(t, T + δ) \]  

where δ stands for a compounding period. Forward LIBOR can be derived via Equation (1) from the prices of zero-coupon bonds.

BGM [1997] modeled interest rate behavior in terms of forward LIBOR based on the arbitrage-free conditions presented in Heath, Jarrow, and Morton [1992]. We briefly specify their results in the following proposition.

Proposition 2.1. LIBOR Dynamics under the Measure Q

The dynamics of LIBOR L(t, T) under the spot martingale measure Q are as follows:

\[ dL(t, T) = \frac{dL(t, T)}{L(t, T)} = \gamma(t, T) - \sigma(t, T + \delta) \delta l(t) + \gamma(t, T) \cdot dW(t) \]  

where 0 ≤ t ≤ T ≤ T and \( \sigma_p(t, \cdot) \) are defined as follows:

\[ \sigma_p(t, T) = \begin{cases} \sum_{j=1}^{m} \frac{\delta L(t, T - j\delta)}{1 + \delta L(t, T - j\delta)} \gamma(t, T - j\delta) & \text{if } t \in [0, T - \delta] \\ 0 & \text{otherwise} \end{cases} \]  

where \( \delta^{-1}(T - t) \) denotes the greatest integer that is less than \( \delta^{-1}(T - t) \), and the deterministic function \( \gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is bounded and piecewise continuous.

According to the derivation process of BGM [1997], \{\sigma_p(t, T)\}_{t\in[0,T]} \) stands for the volatility process of the bond price \( P(t, T) \), which is stochastic rather than deterministic. Thus, the stochastic differential Equation (2) is not solvable, and the distribution of \( L(T, T) \) is unknown. Given a fixed initial time assumed to be zero, however, we can approximate \( \sigma_p(t, T) \) by \( \tilde{\sigma}_p(t, T) \), which is defined by

\[ \tilde{\sigma}_p(t, T) = \begin{cases} \sum_{j=1}^{m} \frac{\delta L(t, T - j\delta)}{1 + \delta L(t, T - j\delta)} \gamma(t, T - j\delta) & \text{if } t \in [0, T - \delta] \\ 0 & \text{otherwise} \end{cases} \]  

where 0 ≤ t ≤ T ≤ T. The calendar time of the process \{\tilde{L}(t, T - j\delta)\}_{t\in[0,T-\delta]} \) in Equation (4) is frozen at its initial time 0, and thus the process \{\tilde{\sigma}_p(t, T)\}_{t\in[0,T]} \) becomes deterministic. By substituting \( \tilde{\sigma}_p(t, T + \delta) \) for \( \sigma_p(t, T + \delta) \) in the drift terms of Equation (2), we can solve it and find the approximate distribution of \( L(T, T) \) to be lognormally distributed.

This technique was first used by BGM [1997] for pricing interest rate swaptions. It was developed further by Brace, Dun, and Barton [1998] and formalized by Brace and Womersley [2000]. It was also used by Schloegl [2002]. This approximation has been widely used in the marketplace and has been shown to be very accurate.

Proposition 2.2. The Lognormalized LIBOR Market Model

The dynamics of lognormalized forward LIBOR \( L(t, T) \) under the spot martingale measure Q are given by
\[
\frac{dL(t, T)}{L(t, T)} = \gamma(t, T) \cdot \overline{\sigma}_0(t, T + \delta) dt + \gamma(t, T) \cdot dW(t)
\]

where \(\overline{\sigma}_0(t, T + \delta)\) is defined in Equation (4).

In pricing interest rate derivatives, it is sometimes more convenient to compute under a specific probability measure rather than the spot martingale measure \(Q\). Thus, we need to know the processes of LIBOR rates under other martingale measures. The following proposition specifies the general rule under which LIBOR dynamics are changed following a change of the underlying probability measure.

**Proposition 2.3.** The Drift Adjustment Technique in Different Measures

The dynamics of forward LIBOR \(L(t, T)\) under an arbitrary forward martingale measure \(Q^f\) are given as follows:

\[
\frac{dL(t, T)}{L(t, T)} = \gamma(t, T) \cdot (\overline{\sigma}_0(t, T + \delta) - \overline{\sigma}_0(t, S)) dt + \gamma(t, T) \cdot dW(t)
\]

where \(0 \leq t \leq \min(S, T)\).5

An Approximate Distribution of a Swap Rate in the LMM Framework

Jamshidian [1997] presented the SMM under which the forward swap rate is a martingale and swap rates can be priced via Black’s formula. It is well known that the LMM and the SMM are incompatible, because swap rates and LIBOR rates cannot be lognormally distributed under the same probability measure. Therefore, choosing either of the two models as a pricing foundation is problematic. Because of mathematical tractability, Brace, Dun, and Barton [1998] suggested that the LMM should be adopted as the central model, and we follow their suggestion.

We chose the LMM to price CMSs. Accordingly, the first problem we encounter is how to find the approximate distribution of swap rates under a forward measure. As we will show later, a swap rate is roughly a weighted average of LIBOR rates, and the LIBOR rates under the LMM are approximately lognormally distributed. As a result, the distribution of a swap rate is roughly a weighted average of lognormal distributions.

This subsection presents a new approach to finding the approximate distribution of a swap rate under the LMM framework.

Define an \(n\)-year forward swap rate, observed at \(t\) and maturing at time \(T_j\) with \(0 \leq t \leq T_j\), as follows:

\[
S_n(t, T_j) = \sum_{j=0}^{\delta-1} w_n^{ij}(t)L(t, T_{ijn})
\]

where

\[
w_n^{ij}(t) = \frac{P(t, T_{ijn})}{\Sigma_{i=0}^{\delta-1} P(t, T_{ijn})}
\]

and \(\{T_1, T_{r1}, ..., T_{rn} \ldots \} \) and \(\{T_{ijn}, T_{ijn+1}, ..., T_{ijn+\delta} \} \) denote, respectively, reset and payment dates with a constant-year fraction (i.e., \(\delta = T_{ijn} - T_{ijn-1}, j = 1, 2, ..., \delta\)). Brace and Womersley [2000] showed that the variability of the \(w_n^{ij}(t)\) was small compared to the variability of forward LIBOR \(L(t, T_{ijn})\), and their conclusion was further empirically confirmed by Brigo and Mercurio [2006]. Thus, we can freeze the value of the process \(w_n^{ij}(t)\) to its initial value, \(w_n^{ij}(0)\), and obtain

\[
S_n(t, T_j) = \sum_{j=0}^{\delta-1} w_n^{ij}(0)L(t, T_{ijn})
\]

By observing Equation (9), \(S_n(t, T_j) \) is a weighted average of lognormally distributed variables, and therefore its distribution is unknown. Although \(S_n(T_j, T)\) is not a lognormal distribution, it can be well approximated by a lognormal distribution with the correct first two moments. Based on this approximation, we assume that \(\ln S_n(T_j, T)\) has a normal distribution with mean \(M\) and variance \(V^2\). The moment-generating function for \(\ln S_n(T_j, T)\) is given by

\[
M_{\ln S}(h) = \mathbb{E}[S_n(T_j, T)^h] = \exp\left(Mh + \frac{1}{2}V^2h^2\right)
\]
where $h = 1$ and $h = 2$ in Equation (10), respectively, we obtain the two conditions to be solved for $M$ and $V^2$ as follows:

$$M = 2 \ln \mathbb{E}[S_n(T, T)] - \frac{1}{2} \ln \mathbb{E}[S_n(T, T)^2]$$

$$V^2 = \ln \mathbb{E}[S_n(T, T)^2] - 2 \ln \mathbb{E}[S_n(T, T)]$$

where $\mathbb{E}[S_n(t, T)]$ and $\mathbb{E}[S_n(t, T)^2]$ are computed in the Appendix.

The accuracy of this technique has been examined by Mitchell [1968]. Furthermore, many areas of science have verified the lognormal approximation to be highly accurate by the sum of lognormal random variables (e.g., Aitchison and Brown [1957], Crow and Shimizu [1988], Levy [1992], Limpert, Stahel, and Abbt [2001], and Borovkova, Permana, and Weide [2007]). In the next to last section, we will provide detailed empirical results to show that this technique is robustly accurate in pricing CMS derivatives.

**PRICING CONSTANT MATURITY SWAPS**

In this section, we use the approximation method from the previous section to price three types of CMSs, namely, CMS-for-CMS, CMS-for-LIBOR, and CMS-for-fixed.

**Pricing CMS-for-CMS**

A CMS-for-CMS constant maturity swap is called a two-way CMS. It is mainly sensitive to the slope of the yield curve and thus can be used to take a profit from a change in the difference between long-term and short-term interest rates. As the yield curve steepens, investors may think that the long-term swap rates will not remain as high in the future as the market reveals, and thus they may take a position in CMSs by paying long-term CMS rates and receiving short-term CMS rates. Alternatively, in a flat-yield-curve market, investors may believe that long-term interest rates may rise in the near future and wish to take a position in CMSs by paying short-term CMS rates and receiving long-term CMS rates.

To provide a general pricing formula for two-way CMSs, we define a general CMS contract as follows.

Consider a $\tau$-year two-way CMS starting at time $T_0$ with payment dates $T_1 \leq T_2 \leq \ldots \leq T_{\tau}$ for $\tau = 1, 2, \ldots, q_\tau = \tau / \delta$, and the notional principal is $\$1$. At each payment date $T_i$ (in some variants at $T_i + 1$) for $i = 1, \ldots, q_\tau$, one party pays (receives) $\delta S_n(T_i, T) - K_i$ to the counterparty and receives (pays) $\delta S_m(T, T_i)$ from the counterparty, where $S_n(T_i, T)$ and $S_m(T, T_i)$ denote, respectively, $n$-year and $m$-year swap rates observed at time $T_i$, and $K_i$ is a premium (or a discount). For the party who pays $\delta S_m(T, T_i)$ and receives $\delta S_n(T_i, T) - K_i$, the cash flow stream is given as follows:

At time $T_1$:

$$\delta \left[ S_n(T_1, T) - \delta S_n(T_1, T) - K_1 \right]$$

At time $T_2$:

$$\vdots$$

At time $T_q$:

$$\delta \left[ S_n(T_q, T) - S_m(T, T_q) - K_q \right]$$

The pricing formula of the two-way CMS is presented in the following theorem, and its proof is provided in the Appendix.

**Theorem 3.1.** The Approximate Pricing Formula of the Two-Way CMSs

For the party who pays $m$-year CMS rates and receives $n$-year CMS rates, the price of the $\tau$-year two-way CMS at time $t$, $T_0 \leq t \leq T_{\tau}$, is given by

$$\text{CMS}(t) = \delta \sum_{j=1}^{q_{\tau}} P(t, T_j) \mathbb{E}^{Q^t} \left[ S_n(T_j, T) | \mathcal{F}_t \right] - \mathbb{E}^{Q^t} \left[ S_m(T, T) | \mathcal{F}_t \right] - K_t$$

where

$$\mathbb{E}^{Q^t} \left[ S_n(T_j, T) | \mathcal{F}_t \right] = \sum_{j=0}^{q_{\tau}} w_{ij}(t) L(t, T_{ij}) \zeta(t, T_{ij}; T_j)$$

$$\zeta(t, T_{ij}; T_j) = \exp \left( \int_{T_{ij}}^{T_j} \Delta(u, T_{ij}; T_j) du \right)$$

$$w_{ij}(t) = \frac{P(t, T_{ij})}{\sum_{i=1}^{q_{\tau}} P(t, T_{ij})}$$

K curve. These features will be examined in the next section.

Equation (13), two-way CMSs are almost immunized from convexity adjustments, and involved in the CMS rate by different convexity adjustments, \( \zeta \). By observing the pricing formula in Equation (13), two-way CMSs are almost immunized from any parallel shift in the interest rate yield curve, but they are sensitive to change in the slope of the interest rate yield curve. These features will be examined in the next section.

As a CMS is priced at contract initiation, \( T_0 \), its price is reflected in the premium \( K_1 \). By adjusting the premium \( K_1 \), the initial price of the two-way CMS can be set to zero and trading becomes a fair game. This fair premium \( K_1 \) is provided by

\[
K_1 = \sum_{i=0}^{q} \omega_{T_i} E^{Q} [S_{n}(T_i, T_{i+1}) | \mathcal{F}_{T_i}] - \sum_{i=0}^{m} \omega_{T_i} E^{Q} [S_{m}(T_i, T_{i+1}) | \mathcal{F}_{T_i}] \tag{20}
\]

where

\[
\omega_{T_i} = \frac{P(T_i, T_{i+1})}{\sum_{j=0}^{k} P(T_{i_j}, T_{i_j+1})}
\]

By observing Equation (20), the fair premium is represented by the difference between two weighted averages of CMS rates over the life of the transaction.

Pricing CMS-for-LIBOR

A CMS-for-LIBOR constant maturity swap is similar to the two-way CMS described in the previous subsection, except that one of the CMS rates is replaced by LIBOR in the cash flow stream. At each payment date \( T_i \) for \( i = 1, \ldots, q \), one party pays (receives) \( \delta(S(T_i, T_{i+1}) - K_2) \) to the counterparty, where \( K_2 \) is a premium (or a discount), and receives (pays) from the counterparty \( \delta L(T_i, T_{i+1}) \) (or \( \delta L(T_{i-1}, T_{i}) \) in some variants) where \( L(T_i, T_{i+1}) \) is LIBOR for the period \([T_i, T_{i+1}]\). For the party who pays \( \delta L(T_i, T_{i+1}) \) and receives \( \delta (S(T_i, T_{i+1}) - K_2) \), the cash flow stream is given as follows:

- At time \( T_1 \) : \( \delta [S(T_1, T_{1+1}) - L(T_1, T_{1+1}) - K_2] \)
- At time \( T_2 \) : \( \delta [S(T_2, T_{2+1}) - L(T_2, T_{2+1}) - K_2] \)
- \( \vdots \)
- At time \( T_q \) : \( \delta [S(T_q, T_{q+1}) - L(T_q, T_{q+1}) - K_2] \)

The pricing formula of the CMS-for-LIBOR CMS is presented in the following theorem.

**Theorem 3.2.** The Approximate Pricing Formula of CMS-for-LIBOR CMSs

The price of the \( T \)-year CMS-for-LIBOR CMS at time \( t \), \( T_0 \leq t \leq T_q \), is given by

\[
CMS(t) = \delta \sum_{i=0}^{q} P(t, T_i) E^{Q} [S_{n}(T_i, T_{i+1}) | \mathcal{F}_{t}] - L(t, T_i) E^{Q} [S_{m}(T_i, T_{i+1}) | \mathcal{F}_{t}] - K_2 \tag{21}
\]

where \( E^{Q} [S_{n}(T_i, T_{i+1}) | \mathcal{F}_{t}] \) and \( E^{Q} [S_{m}(T_i, T_{i+1}) | \mathcal{F}_{t}] \) are defined, respectively, in Equations (14) and (15), and \( L(t, T_i) \) denotes the forward LIBOR rate that is observable from market data.8

It is well known that a one-period swap rate is actually a LIBOR rate, and thus a CMS-for-LIBOR CMS is a special case of a two-way CMS. As the tenor, \( m \), of the paid swap rate \( S_{m}(T_i, T_{i+1}) \) is set to \( \delta \), Equation (13) reduces to Equation (21). Similar to Equation (13), CMS-for-LIBOR CMSs are sensitive to a change in the slope of the interest rate yield curve, but almost immunized from any parallel shift in the interest rate yield curve.

End-users usually use CMS-for-LIBOR CMSs to adjust their asset or liability duration. For example, life insurers may be indebted to long-period payment obligations, but own short-duration assets (may be receiving LIBOR payments). To remove interest rate risk, they may use CMS-for-LIBOR CMSs to transform their LIBOR...
The fair premium $K_2$ can be regarded as the difference between the weighted averages of CMS rates and LIBOR rates over the life of the transaction.

### Pricing CMS-for-Fixed

A CMS-for-fixed constant maturity swap is similar to the two-way CMS described in the subsection on pricing CMS-for-CMS CMSs, except that one of the CMS rates is replaced by a fixed-rate $R$ in the cash flow stream. At each payment date $T_i$ for $i = 1, \ldots, q$, one party pays (receives) $\delta S_n(T_i, T)$ to the counterparty and receives (pays) from the counterparty $\delta R$. For the party who pays $\delta R$ and receives $\delta S_n(T_i, T)$, the cash flow stream is given as follows:

- At time $T_1$: $\delta [S_n(T_1, T_1) - R]$
- At time $T_2$: $\delta [S_n(T_2, T_2) - R]$
- \hspace{1cm} \vdots
- At time $T_q$: $\delta [S_n(T_q, T_q) - R]$

The pricing formula of the CMS-for-fixed CMSs is presented in the following theorem.

**Theorem 3.3.** The Approximate Pricing Formula of CMS-for-Fixed CMSs

The price of the $\tau$-year CMS-for-fixed CMS at time $t$, $T_0 \leq t \leq T_1$, is given by

$$CMS_n(t) = \delta \sum_{i=1}^{q} P(t, T_i) E_{\mathcal{F}_t}^{\Omega} \left[ S_n(T_i, T_1) \mid \mathcal{F}_t \right] - R$$

where $E^{\Omega} \left[ S_n(T_i, T_1) \mid \mathcal{F}_t \right]$ is defined in Equation (14).

As the paid swap rate $S_n(T_i, T)$ degenerates to a constant rate $R$, Equation (13) reduces to Equation (23), and thus a CMS-for-fixed CMS is also a special case of a two-way CMS. Likewise, CMS-for-fixed CMSs are also sensitive to change in the slope of the interest rate yield curve. Unlike the aforementioned CMSs, CMS-for-fixed CMSs are sensitive to any parallel shift in the interest rate yield curve. These features will also be examined in the next section.

The prices of CMS-for-fixed CMSs are reflected in the fixed rates. By adjusting $R$, the initial price of the CMS can be set to zero and the fair rate $R$ is solved as

$$R = \frac{1}{\delta} \sum_{i=1}^{q} P(t, T_i) E_{\mathcal{F}_t}^{\Omega} \left[ S_n(T_i, T_1) \mid \mathcal{F}_t \right]$$

The fair rate $R$ is thus the weighted average of expected CMS rates over the life of the transaction.

Corporations and financial institutions usually employ CMS-for-fixed CMSs to enhance yields. Investors who receive fixed-rate cash inflows, and expect the back end of the yield curve to rise steeply, may enter into this CMS to take profits by receiving long-term CMS rates while paying fixed rates.

We have provided the pricing formulas of three types of CMSs by employing the approximation methods introduced in the second section. The accuracy of these pricing formulas will be examined in the next section by comparison with Monte Carlo simulations.

### NUMERICAL STUDIES

This section first presents a method to calibrate the parameters in the LMM and then provides some numerical examples to examine the accuracy of our pricing formula. In the last subsection, we examine the sensitivity of CMSs to changes in yield curves, volatilities, and times to maturity.
Calibration Procedure

The most important advantages of the LMM are its tractability and feasibility. The cap and floor pricing formulas within the LMM framework are Black’s formulas, and thus the model volatilities can be extracted directly from quoted implied volatilities of caps. Since the standard pricing formula of caplets involves only a single forward LIBOR rate, the correlation matrix of the forward LIBOR rates cannot be extracted from quoted cap prices. There are two main methods widely used in the marketplace to calibrate the correlation matrix between LIBOR rates. The first method is based on the price quotations of swaptions, and the second is presented by Rebonato [1999], who applied a historical correlation matrix to engage in a simultaneous calibration of the LMM to the volatilities and correlation matrix of the forward LIBOR rates. This method is also adopted by Wu and Chen [2007a, 2007b] to price equity swaps.

We briefly introduce the method as follows. Assume there are \( n \) forward LIBOR rates in the \( m \)-factor framework. Assume that each forward LIBOR rate, \( L(\cdot, T_i) \), has a constant instantaneous volatility, namely, for \( i = 1, \ldots, n \), \( \gamma(\cdot, T_i) = \nu_i \). The setting is presented in Exhibit 2. Thus, if the market-quoted volatility for the \( T_i \)-year cap is \( \xi_i \), then \( \nu_i = \xi_i \). Next, for \( i = 2, \ldots, n \), if the \( T_i \)-year cap is \( \xi_i \), then \( \nu_i = \xi_i T_i^2 - \xi_{i-1} T_{i-1}^2 \).

In addition, we use the historical data of the forward LIBOR rates to derive a market correlation matrix \( \Phi \). \( \Phi \) is an \( n \)-rank, positive-definite, and symmetric matrix that can be written as

\[
\Phi = H \Gamma H
\]

where \( H \) is a real orthogonal matrix and \( \Gamma \) is a diagonal matrix. Let \( A = H^{-1/2} \), and thus \( \Phi = AA' \). In this way, we can find an \( m \)-rank \( (m \leq n) \) matrix \( B \) so that \( \Phi^B = BB' \) is an approximate correlation matrix for \( \Phi \).

The advantage of finding \( B \) is that we may replace the \( n \)-dimensional original Brownian motion \( dZ(t) \) with \( BdW(t) \) where \( dW(t) \) is an \( m \)-dimensional Brownian motion such that an approximate correlation structure is given by

\[
BdW(t) (BdW(t))' = BdW(t) dW(t) B' = BB' dt = \Phi^B dt
\]

We must still find a suitable matrix \( B \). Rebonato [1999] proposed a method to find \( B \), which is described as follows. Assume that the \( ik \)th element of \( B \), for \( i = 1, 2, \ldots, n \), is specified as

\[
b_{ik} = \begin{cases} \cos \theta_{ik} \Pi_{j=1 \atop k \neq j}^{i-1} \sin \theta_{ij} & \text{if } k = 1, 2, \ldots, m-1 \\ \Pi_{j=1 \atop k \neq j}^{i-1} \sin \theta_{ij} & \text{if } k = m \end{cases}
\]

Thus, \( \Phi^B = BB' \) is a function of \( \Theta = \{ \theta_{i,k} \}_{i=1 \atop k=1}^n \). An optimal solution \( \hat{\Theta} \) is obtained by solving the following optimization problem:

\[
\min_{\Theta} \sum_{i,j=1}^{n} | \Phi_{ij}^B - \Phi_{ij} |^2 \tag{25}
\]

where \( \Phi_{ij}^B \) is the \( ij \)th element of \( \Phi^B \) and \( \Phi_{ij} \) is the \( ij \)th element of \( \Phi \), specifically defined as follows:

\[
\Phi_{ij} = \sum_{k=1}^{m} b_{ik} b_{kj} \]

By substituting \( \hat{\Theta} \) into \( B \), we obtain an optimal matrix \( \hat{B} \) such that \( \Phi_{ij}^B (= \hat{BB}') \) is an approximate correlation matrix for \( \Phi \).

We use \( \hat{B} \) to distribute the instantaneous total volatility, \( \nu_i \) to each Brownian motion without changing the amount of the instantaneous total volatility. That is,

### Exhibit 2

**Instantaneous Volatilities of \( L(t, \cdot) \)**

| Instant. Total Vol. | Time \( t \in (T_{n-2}, T_{n-1}) \) | \( T_{n-2}, T_{n-1} \) | \( (T_{n-2}, T_{n-1}) \) | \( (T_{n-2}, T_{n-1}) \) |
|---------------------|---------------------------------|------------------------|------------------------|
| Fwd Rate: \( L(t, T_1) \) | \( v_1 \) | Dead | Dead | ... | Dead |
| \( L(t, T_2) \) | \( v_2 \) | \( v_2 \) | Dead | ... | Dead |
| ... | ... | ... | ... | ... | ... |
| \( L(t, T_n) \) | \( v_n \) | \( v_n \) | \( v_n \) | ... | \( v_n \) |
The CMS rates are reset semiannually, and the premium CMS rates with times to maturity of 1, 3, 5, and 7 years. CMSs receiving 10-year CMS rates and paying 2-year...

In addition, we also provide a comparison of our approximation formula with the convexity adjustment formulas introduced by Brigo and Mercurio [2006]. Brigo and Mercurio introduced two convexity adjustment formulas in their Equations (13.15) and (13.16) that are widely used in the marketplace. We find that one of the two formulas always slightly overestimates, and the other slightly underestimates. Therefore, we use an average of the two, which is more stable, as a proxy of the convexity adjustment approach.

The pricing results are listed in Exhibit 3, which shows that our pricing formulas are sufficiently accurate and robust in all market scenarios examined as compared with the Monte Carlo simulation. Without closed-form pricing formulas for CMSs, market practitioners would resort to time-consuming Monte Carlo simulation. In contrast, our approximate pricing formulas provide sufficiently accurate prices with significant time savings, which is an important advantage, especially in a competitive financial market. Moreover, our pricing formulas yield values very close to those computed from the convexity adjustment approach, but in comparison with Monte Carlo simulations our pricing errors are less than in the convexity adjustment approach, particularly for long-term CMSs.

Our pricing approach has another significant merit in that it can derive the approximate distribution of swap rates, the convexity adjustment approach only provides the first moment of swap rates. Therefore, our approach can also be used to price options written on swap rates. With our approximation approach, both LIBOR–type and swap–type interest rate derivatives can be priced consistently within the LMM framework. Because of the versatility of the LMM framework, the risk of interest rate derivatives can be managed efficiently and consistently.

Property of CMSs

We have previously mentioned that two-way and CMS–for–LIBOR CMSs are sensitive to change in the slope of the interest rate yield curve but are almost immune to any parallel shift in the curve. CMS–for–fixed CMSs, however, are sensitive to both a change in the slope and parallel shift in the interest rate yield curve. In this subsection, we provide some market scenarios to examine these conclusions.

Consider the three scenarios of initial forward LIBOR curves in Exhibit 4: curves 1 and 2 are flat curves, namely, 2% and 8%, and curve 3 is spread uniformly between 2% and 8%. The difference between curves 2 and 1 stands for a parallel shift in the curve; the difference between curves 3 and 1 stands for a change in the slope of the yield curve. In addition, we consider three flat
volatility levels, namely, 10%, 20%, and 30%, and CMS times to maturity from 1 year to 10 years. The CMSs considered in this subsection are the two-way CMS receiving 10-year CMS rates and paying 2-year CMS rates, the CMS-for-LIBOR CMS receiving 10-year CMS rates, and paying 6-month LIBOR rates, and the CMS-for-fixed CMS receiving 10-year CMS rates and paying a fixed rate.13

For each volatility level and time to maturity, we first compute the prices of two-way CMSs via the pricing formulas in Theorem 1 by using the initial forward LIBOR curves 1, 2, and 3, respectively, and then compute the price differences of the two-way CMSs calculated by using curves 2 and 1 and the price differences by using curves 3 and 1. Similarly, we compute the price differences of CMS-for-LIBOR and CMS-for-fixed CMSs.

Exhibit 5 shows the results. The first row in Exhibit 5 presents price differences using curves 2 and 1, and the second row presents price differences using curves 3 and 1. The first column in Exhibit 5 presents price differences using flat volatility 10 and the other columns are 20 and 30, respectively. Each figure contains three types of CMS price differences.

As anticipated, Exhibit 5 indicates that two-way and CMS-for-LIBOR CMSs are sensitive to change in the slope of the interest rate yield curve but are almost immunized from any parallel shift in the curve. CMS-for-fixed...
CMSs, however, are sensitive to both change in the slope and any parallel shift in the interest rate yield curve. These effects become more profound as the time to maturity and the volatility level increase.

**CONCLUSION**

We provide a technique to approximate the distribution of forward swap rates under the LMM and use it to price three types of CMS contracts, namely, CMS-for-CMS, CMS-for-LIBOR, and CMS-for-fixed. The resulting pricing formulas are shown to be sufficiently accurate by comparison with Monte Carlo simulation. We also examine some properties of CMSs with respect to initial LIBOR zero curves, volatility levels, and times to maturity. Pricing CMSs via our pricing formulas is time saving and accurate and provides a good alternative to time-consuming Monte Carlo simulation. Therefore, the pricing models are worth recommending to market practitioners.

**APPENDIX**

The Proof of Theorem 1

Based on the risk-neutral valuation, the time-$t$ price of the two-way CMS can be computed under forward martingale measure $Q^\gamma$ as follows:

\[
CMS(t) = \delta \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} P(t,T_j) \mathbb{E}^{Q^\gamma}[S(T_i,T_j) | \mathcal{F}_t] - \mathbb{E}^{Q^\gamma}[S(0,T_i) | \mathcal{F}_t] - K_i
\]  

(A-1)

Since the derivation processes of the two expectations in Equation (A-1) are the same, we compute only the first expectation; the second can be computed similarly.

According to Equation (9), the first expectation can be derived as follows:

\[
\mathbb{E}^{Q^\gamma}[S(0,T_i) | \mathcal{F}_t] = \sum_{i=1}^{\infty} w^{(0)}(t) \mathbb{E}^{Q^\gamma}[L(t,T_i) | \mathcal{F}_t]
\]  

(A-2)

where $w^{(0)}(t)$ is frozen to its initial value, $w^{(0)}(t)$. According to Proposition 2.3, under forward martingale measure $Q^\gamma$, the dynamics of $L(\cdot, T_{i+1})$ are given as follows:

\[
\frac{dL(u,T_{i+1})}{L(u,T_{i+1})} = \gamma(u,T_{i+1}) \cdot (\overline{\sigma}(u,T_{i+1}) - \underline{\sigma}(u,T)) dt + \gamma(u,T_{i+1}) \cdot dW(u)
\]

and via Itô’s lemma, we have

\[
L(T_i,T_{i+1}) = L(t,T_i) \exp \left( \int_t^T \Delta(u,T_{i+1};T) du - \frac{1}{2} \left[ \int_t^T \left( \left| \gamma(u,T_{i+1}) \right| \right)^2 du + \int_t^T \gamma(u,T_{i+1})^2 du \right] \right)
\]  

(A-3)

where

\[
\Delta(u,T_{i+1};T) = \gamma(u,T_{i+1}) \cdot (\overline{\sigma}(u,T_{i+1}) - \underline{\sigma}(u,T))
\]

Inserting Equation (A-3) into Equation (A-2), we have

\[
\mathbb{E}^{Q^\gamma}[S(0,T_i) | \mathcal{F}_t] = \sum_{j=0}^{\infty} w^{(0)}(t) L(t,T_i) \zeta(t,T_{i+1};T_i)
\]

where

\[
\zeta(t,T_{i+1};T_i) = \exp \left( \int_t^T \Delta(u,T_{i+1};T) du \right)
\]

**EXHIBIT 4**

Three Scenarios of LIBOR Zero Curves

Note: Curves 1 and 2 are flat curves, namely, 2% and 8%; curve 3 is uniformly spread between 2% and 8%.
E X H I B I T 5

CMS Price Change Relative to Change of Yield Curve. Time to Maturity, and Volatility

Note: The first row presents price differences using curves 2 and 1 and the second row presents price differences using curves 3 and 1. The first column presents price differences using a flat volatility of 10, and the second and third columns use volatilities of 20 and 30, respectively. Each figure contains three types of CMS price differences.

ENDNOTES

1As examined by Rogers [1996], the Gaussian term structure model has an important theoretical limitation: the rate can attain negative values with positive probability, which may cause pricing errors in many cases.

2The LMM has been adopted to price many exotic interest rate options, (e.g., Wu and Chen [2008, 2009a, 2009b], and Benner, Zyapkov, and Jortzik [2009]).

3The filtration \( \{ \mathcal{F}_t \}_{t \in [0,T]} \) is right continuous, and \( \mathcal{F}_0 \) contains all the \( Q \)-null sets of \( \mathcal{F} \).

4For ease of computation in Equation (3), \( \delta \) may be fixed (for example, \( \delta = 0.5 \)).

5We employ \( W(t) \) to denote an independent \( m \)-dimensional standard Brownian motion under an arbitrary measure without causing any confusion.

6Brace and Womersley [2000] also showed that \( w_{\text{log}}(t) \) is a martingale under some probability measure. Therefore, it is not unreasonable to approximate \( w_{\text{log}}(t) \) with its initial value \( w_{\text{log}}(0) \).

7As indicated by Brigo and Mercurio [2006], forward swap rates obtained from lognormal forward LIBOR rates are not far from being lognormal under the relevant measure.

8If the LIBOR payment is paid in arrears, the term \( L(t, T_j) \) \( \zeta(t, T_j, T_k) \) in Equation (21) becomes \( L(t, T_{j-1}) \).

9For details about the first method, please refer to Brigo and Mercurio [2006].

10For other assumptions of volatility structures, please refer to Brigo and Mercurio [2006].

11Note that the Euclidean norm of each row vector of \( B \) is one.
For more details regarding the performance of single- and multifactor models, please refer to Driessen, Klaassen, and Melenberg [2003] and Rebonato [1999].

The value of the fixed rate has no impact on the result, and therefore we do not specify it.

REFERENCES


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