Currency-Protected Swaps and Swaptions with Nonzero Spreads in a Multicurrency LMM

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Despite the fact that currency-protected swaps and swaptions are widely traded in the marketplace, pricing models for zero-spread swaps, and swaptions have rarely been examined in the extant literature. This study presents a multicurrency LIBOR market model and uses it to derive pricing formulas for currency-protected swaps and swaptions with nonzero spreads. The resulting pricing formulas are shown to be feasible and tractable for practical implementation and their hedging strategies are also provided. Our pricing formulas provide prices close to those computed from Monte Carlo simulation, but involve far less computation time, and thereby offering almost instant price quotes to clients and daily marking-to-market trading books, and facilitating efficient risk management of trading positions. © 2012 Wiley Periodicals, Inc. Jrl Fut Mark

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1. INTRODUCTION

A currency-protected swap (CPS), also called a cross-currency basis swap, (LIBOR) differential swap or interest rate index swap, is a variation of a plain-vanilla interest rate swap. A CPS involves the exchange of two series of floating payments, both denominated in the same currency, while reference rates are LIBOR rates in two different currencies. A nonzero spread is often contracted in a CPS to compensate for the difference between two interest rates and the relative credit ratings of two counterparties. Just like ordinary interest rate swaps, CPSs are normally settled in arrears, and there is no exchange of principal.

The first CPS was traded in late 1990 between Credit Suisse First Bank and a Japanese insurance company, and the instrument has been widely traded since then.\(^1\) The main impetus to the growth of the CPS market lies in a difference in the relation between global term structures of interest rates; the differences in long-term interest rates among a number of currencies fail to reflect the differentials among their short-term rates. For example, a considerable number of CPSs were traded in the period between the second half of 1991 and early 1992 due to the fact that some interest rate term structures in the global market were upward sloping (e.g., in the United States), while others were flat or downward sloping (primary in Europe), which led to the accelerating growth of the CPS market.

The major use of CPSs is to exploit the floating rate difference or the money market interest rate difference between two currencies without incurring direct exchange rate risk. For example, a portfolio manager can employ CPSs to enhance current yield if he can take a correct view on the movement of interest rate differentials between two currencies. In currencies with relatively high interest rates, borrowers may enter into CPSs to reduce interest costs by capitalizing on the interest rate differential between two currencies. In addition, corporate treasurers may use CPSs to transform their yields (from assets) or costs (from liabilities) to link interest rates in one currency with those in another favorable currency, while payments are still made in domestic currency.

Currency-protected swaptions (CPSOs), which provide their owners the right to enter or terminate CPSs, have also been traded in over-the-counter markets. An investor who wants to take a position in a CPS in the future and finds that the current CPS price is attractive, can lock in the CPS price via a CPSO.

To present, few research papers have examined the pricing and hedging of CPSs and CPSOs. Litzenberger (1992) first discussed CPSs and proposed some analytical pricing methods in his presidential address. Turnbull (1993) adopts

\(^1\) According to the data from BIS, the outstanding notional principal of CPSs is amounted to about $37 billion in 2007.
the cross-currency Gaussian Heath, Jarrow, and Morton (1992, HJM) model to price and hedge CPSs. Jamshidian (1994) examines the pricing and hedging of CPSs in a replication approach by using domestic and foreign zero-coupon bonds (ZCB), but only a general expression of the solution is provided, which cannot be used without further analysis. Both Wei (1994) and Chang, Chung, and Yu (2002) employ an Ornstein–Uhlenbeck process to specify the behavior of domestic and foreign risk-free interest rates and used the resulting model to price CPSs. The studies cited above provide theoretically sound pricing models for CPSs. Unfortunately, they are intractable in parameter calibration and thus not well suited for practical implementation. Moreover, the rates specified in traditional interest rate models, such as Vasicek (1977) and Gaussian HJM (1992), can yield negative values with positive probability, which may result in pricing errors in many cases.2

To overcome the obstacles in parameter calibration of traditional interest rate models, the LIBOR market model (LMM) is thus developed by Brace, Gatarek, and Musiela (1997), Musiela and Rutkowski (1997), and Miltersen, Sandmann, and Sondermann (1997). The rates specified in the LMM are market-observable LIBOR rates, which have been used as underlying interest rates in many popular financial instruments. The LIBOR rate modeled in the LMM is lognormally distributed, thereby avoiding the pricing errors resulting from possible negative interest rates. The LMM can be used to price most interest rate derivatives, and thus the interest rate risks can be consistently managed.3 In addition, the most important merits of the LMM are its feasibility and tractability in parameter calibration. The LMM can simultaneously calibrate market-quoted cap volatilities and the correlation matrix of underlying forward LIBOR rates.

Due to the advantages presented above, Schlögl (2002) adopted a cross-currency LMM to price CPSs and CPSOs. However, the spread of CPSs and CPSOs in Schlögl (2002) is set to zero, and thus their pricing formulas cannot be used to price nonzero-spreads CPSs and CPSOs that are more popularly traded in the marketplace. In addition, by following interest rate parity, there is a concrete relation among domestic and foreign forward LIBOR rates and the forward exchange rate, so are their volatilities. Therefore, Schlögl (2002) imposes a restriction on the choice of lognormally distributed underlying variables. He then derives a CPSO pricing formula under the assumption that the domestic forward LIBOR rate and forward exchange rate volatilities are deterministic.4 However, under the model setting of Schlögl (2002), the parameter

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2 Please refer to Rogers (1996) for the negative interest rate problem.
3 The LMM are also adopted to price exotic interest rate swaps and options. Please refer to Wu and Chen (2009a, 2010).
4 Refer to Schlögl (2002) for a detailed discussion.
calibration of the resulting CPSO pricing formula is not easy to carry out in practice.

Based on the arbitrage-free conditions in Amin and Jarrow (1991), the present study extends the LMM to a multicurrency LIBOR market model (MLMM), and then adopts it to examine the pricing and hedging of CPSs and CPSOs. It assumes the domestic and foreign forward LIBOR rate volatilities are deterministic, which makes parameter calibration more tractable. By extending the pricing formulas in Schlögl (2002), our pricing formulas can be used to price nonzero-spread CPSs and CPSOs that are frequently traded in the marketplace. In addition, we further examine the pricing and hedging of a variant CPS, which exchanges interest rates in one foreign currency to another foreign currency, both denominated in domestic currency. Variant CPSs are also popular in the marketplace due to its great versatility, and its options are also examined in this analysis.

The remainder of this study is organized as follows. In Section 2, we introduce the MLMM and its associated implementation techniques in practice. A generalized lognormal distribution is also introduced in this section. In Section 3, we develop the pricing and hedging of CPSs. The pricing and hedging of CPSOs are presented in Section 4. In Section 5, we provide numerical studies to examine the accuracy of our pricing formulas. In Section 6, we present our conclusions.

2. THE MODEL

In Section 2.1, we briefly review the MLMM and various approximate techniques that are widely used in practical operation. In Section 2.2, we present the approximate dynamics of swap rates within the MLMM framework. In Section 2.3, we introduce a generalized lognormal distribution, which is used to approximate the distribution of the difference between domestic and foreign swap rates.

2.1. Review of the MLMM and Approximate Techniques

Assume that trading takes place continuously in time over an interval \([0, \tau]\), \(0 < \tau < \infty\). The uncertainty is described by the filtered probability space \((\Omega, \mathcal{F}, \mathbb{Q}, \{\mathcal{F}_t\}_{t \in [0, T]})\), where the filtration is generated by an independent

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5 Our MLMM involves interest rates in the environment with the domestic and two various foreign currencies, while Schlögl (2002) specifies interest rates in the domestic and only one foreign currency.

6 Both domestic and foreign caps (floors) are the most liquid source of interest rate volatility information, and their maturity range extends to as far as 20 years.
$m$-dimensional standard Brownian motion $\tilde{W}(t) = (W_1(t), W_2(t), \ldots, W_m(t))$ and $Q$ represents the domestic spot martingale probability measure. We list the following notations with the subscript “d” representing domestic country; “f” and “g” two other foreign countries:

$L_k(t, T) = $ the $k$th country’s forward LIBOR rate contracted at time $t$ for borrowing and lending during the time period $[T, T + \delta]$, where $0 \leq t \leq T < T + \delta \leq \tau$ and $k \in \{d, f, g\}$.

$B_k(t, T) = $ the time $t$ price of the $k$th country’s ZCB expiring and paying one dollar at time $T$.

$r_k(t) = $ the $k$th country’s risk-free short rate at time $t$.

$X_k(t) = $ the spot exchange rate at time $t$ for one unit of the $k$th country’s currency in terms of domestic currency.

$Q^T = $ the domestic forward martingale measure with respect to the numéraire $B_d(\cdot, T)$.

The relationship between LIBOR rates and ZCB prices can be specified as follows:

\[
1 + \delta L_k(t, T) = \frac{B_k(t, T)}{B_k(t, T + \delta)}, \quad k \in \{d, f, g\}.
\] (1)

Based on the arbitrage-free conditions in Amin and Jarrow (1991), we extend the LMM to the MLMM. The result is presented in the following proposition.7

**Proposition 1:** MLMM under domestic spot martingale measure $Q$. Under the domestic spot martingale measure $Q$, the processes of the forward LIBOR rates, the ZCB prices and the spot exchange rates are given as follows:

\[
\frac{dL_d(t, T)}{L_d(t, T)} = \gamma_d(t, T) \cdot \sigma_{B_d}(t, T + \delta) \cdot dt + \gamma_d(t, T) \cdot d\tilde{W}(t),
\] (2)

\[
\frac{dB_d(t, T)}{B_d(t, T)} = r_d(t) \cdot dt - \sigma_{B_d}(t, T) \cdot d\tilde{W}(t),
\] (3)

\[
\frac{dL_k(t, T)}{L_k(t, T)} = \gamma_k(t, T) \cdot (\sigma_{B_k}(t, T + \delta) - \sigma_{X_k}(t)) \cdot dt + \gamma_k(t, T) \cdot d\tilde{W}(t), \quad k \in \{f, g\},
\] (4)

7The derivation procedure can be obtained from Wu and Chen (2007a).
where $\cdot$ stands for the inner product of vectors. $m$-dimensional volatility vector functions, $\gamma_k(t, T)$, $k \in \{d, f, g\}$, and $\sigma_{X_k}(t)$, $k \in \{f, g\}$, are deterministic and satisfy the standard regular conditions. The bond volatility vector function, $\sigma_{B_k}(t, T)$, $k \in \{d, f, g\}$, is defined as follows:

$$
\sigma_{B_k}(t, T) = \begin{cases} 
\delta_{L_k}(t, T - j\delta) \gamma_k(t, T - j\delta) & t \in [0, T - \delta] \\
0 & \text{otherwise},
\end{cases}
$$

where $[\delta^{-1}(T - t)]$ denotes the greatest integer that is less than $\delta^{-1}(T - t)$ and $t \in [0, T]$.

Benner, Zyapkov, and Jortzik (2009, BZJ) also propose another approach to specify the stochastic processes, whose result is equivalent to Proposition 1 and is presented in Proposition 2. Proposition 1 carries simpler notations, and thereby greatly facilitating the pricing process and achieving more elegant pricing formulas. Therefore, the following pricing models are derived based on Proposition 1.

**Proposition 2**: BZJ (2009) Approach for the MLMM under $Q$. Under $Q$, the processes of the forward LIBOR rates, the ZCB prices, and the spot exchange rates are given as follows:

$$
\frac{d L_d(t, T)}{L_d(t, T)} = \rho_{L_d(T), B_d(T)} V_{L_d}(t, T) V_{B_d}(t, T) dt + V_{L_d}(t, T) d Z_{L_d(T)}(t),
$$

$$
\frac{d B_d(t, T)}{B_d(t, T)} = \rho_d(t) dt - V_{B_d}(t, T) d Z_{B_d(T)}(t),
$$

$$
\frac{d L_k(t, T)}{L_k(t, T)} = (\rho_{L_k(T), B_k(T)} V_{L_k}(t, T) V_{B_k}(t, T) - \rho_{L_k(T), X_k} V_{L_k}(t, T) V_{X_k}(t, T)) dt + V_{L_k}(t, T) d Z_{L_k(T)}(t),
$$

$$
\frac{d B_k(t, T)}{B_k(t, T)} = (\rho_k(t) + \rho_{B_k(T), X_k} V_{B_k}(t, T) V_{X_k}(t, T)) dt - V_{B_k}(t, T) d Z_{B_k(T)}(t),
$$
\[
\frac{dX_k(t)}{X_k(t)} = (r_d(t) - r_k(t))dt + V_{X_k}(t)dZ_{X_k}(t),
\]

where

\[
dZ_{Ld(T)}(t) = \gamma_d(t, T) \cdot \frac{d\tilde{W}(t)}{V_{Ld}(t, T)}, \quad dZ_{Bd(T)}(t) = \frac{\sigma_{B_d}(t, T) \cdot d\tilde{W}(t)}{V_{Bd}(t, T)},
\]

\[
dZ_{Lk(T)}(t) = \gamma_k(t, T) \cdot \frac{d\tilde{W}(t)}{V_{Lk}(t, T)}, \quad dZ_{Bk(T)}(t) = \frac{\sigma_{B_k}(t, T) \cdot d\tilde{W}(t)}{V_{Bk}(t, T)},
\]

\[
dZ_{X_k}(t) = \frac{\sigma_{X_k}(t) \cdot d\tilde{W}(t)}{V_{X_k}(t)},
\]

and

\[
\rho_{Ld(T), Bd(T)}dt = dZ_{Ld(T)}(t)dZ_{Bd(T)}(t) \cdot V_{Ld}(t, T)V_{Bd}(t, T), \quad \rho_{Lk(T), Bk(T)}dt = dZ_{Lk(T)}(t)dZ_{Bk(T)}(t) \cdot V_{Lk}(t, T)V_{Bk}(t, T),
\]

\[
\rho_{Lk(T), Xk}dt = dZ_{Lk(T)}(t)dZ_{Xk(T)}(t) \cdot V_{Lk}(t, T)V_{Xk}(t, T), \quad \rho_{Bk(T), Xk}dt = dZ_{Bk(T)}(t)dZ_{Xk(T)}(t) \cdot V_{Bk}(t, T)V_{Xk}(t, T),
\]

and \(V_{L*}(t, T) = \|\gamma_*(t, T)\|, V_{B*}(t, T) = \|\sigma_*(t, T)\|\) and \(V_{X*}(t, T) = \|\sigma_*(t, T)\|, \quad * \in \{d, f, g\}\).

The interest rates specified in the MLMM are market-observable LIBOR rates. Moreover, the cap and floor pricing formulas within the MLMM framework are the Black’s formulas, which are widely used for price quotations in the marketplace. Therefore, the model parameters, including initial LIBOR rates, \(L_k(0, \cdot)\), and the volatilities, \(\gamma_k(t, T)\), can be easily extracted from market data, and \(\sigma_{Bk}(t, T)\) can be calculated from (7). The volatilities of exchange rates can be obtained either from historical data of exchange rates or implied volatilities of currency options. Thus, the parameters in the MLMM can be easily calibrated using market data. This feature makes pricing formulas derived within the MLMM framework more tractable and feasible in practice.

Equation (7) indicates that \(\sigma_{Bk}(t, T + \delta)\) is composed of the LIBOR rate processes, and thus it is stochastic rather than deterministic, thereby causing Equations (2) and (4) to be insolvable and the distribution of \(L_k(T, T)\) is unknown. However, we can approximate \(\sigma_{Bk}(t, T)\) by \(\sigma^0_{Bk}(t, T)\), which is defined
by

\[
\tilde{\sigma}_B^0(t, T) = \begin{cases} 
\delta L_k(0, T - j\delta) & \text{for } j \in \mathbb{Z}, \\
\sum_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \frac{\delta L_k(0, T - j\delta)}{1 + \delta L_k(0, T - j\delta)} \gamma_k(t, T - j\delta), & t \in [0, T - \delta] \\
0 & \text{otherwise},
\end{cases}
\]  

(8)

where the calendar time of the process \(L_k(t, T - j\delta)\) in (7) is frozen at its initial time 0 and thus the process \(\tilde{\sigma}_B^0(t, T)\) becomes deterministic. By substituting \(\tilde{\sigma}_B^0(t, T + \delta)\) for \(\sigma_B(t, T + \delta)\) in the drift terms of (2) and (4), we can compute \(L_k(T, T)\), which follows approximately a lognormal distribution.

This approximate technique is also used in BGM (1997), Schlögl (2002), and Wu and Chen (2007), and has been widely employed in practice due to its accuracy. The result of the lognormalization is concluded in the following proposition.

**Proposition 3:** Approximate LIBOR rate dynamics under \(Q\). The approximate dynamics of forward LIBOR rates under \(Q\) are given as follows:

\[
\frac{dL_d(t, T)}{L_d(t, T)} = \gamma_d(t, T) \cdot \tilde{\sigma}_B^0(t, T + \delta) dt + \gamma_d(t, T) \cdot d\tilde{W}(t),
\]

(9)

\[
\frac{dL_k(t, T)}{L_k(t, T)} = \gamma_k(t, T) \cdot (\tilde{\sigma}_B^0(t, T + \delta) - \sigma_X(t)) dt + \gamma_k(t, T) \cdot d\tilde{W}(t),
\]

(10)

where \(k \in \{f, g\}\) and \(0 \leq t \leq T \leq \tau\).

Occasionally, it is more convenient to compute expected cash flows in different forward martingale measures. The following proposition specifies the general rule under which the LIBOR rate dynamics change following a change in the underlying measure. This rule is useful for deriving the pricing formulas of interest rate derivatives.\(^8\)

**Proposition 4:** Drift adjustment technique in different measures. The dynamics of forward LIBOR rates under an arbitrary domestic forward martingale measure \(Q^{T'}\) is given as follows:

\[
\frac{dL_d(t, T)}{L_d(t, T)} = \gamma_d(t, T) \cdot (\tilde{\sigma}_B^0(t, T + \delta) - \tilde{\sigma}_B^0(t, T')) dt + \gamma_d(t, T) \cdot d\tilde{W}(t),
\]

(11)

\(^8\)For the pricing technique associated with changing measure, please refer to Sundaram (1997) and Benninga, Björk, and Wiener (2002).
\[
\frac{dL_k(t, T)}{L_k(t, T)} = \gamma_k(t, T) \cdot (\hat{\sigma}^0_B(t, T + \delta) - \hat{\sigma}^0_B(t, T') - \sigma_{Xk}(t)) dt + \gamma_k(t, T) \cdot d\tilde{W}(t),
\]

where \( k \in \{f, g\} \) and \( 0 \leq t \leq \min(T, T'). \)

2.2. Approximate Dynamics of Swap Rates
Within the MLMM Framework

The swap market model (SMM) and the LMM are popular in practice due to their agreement with Black’s formulas for caps and swaptions. However, it is well known that the LMM and the SMM are incompatible, and thus choosing either one of the two models as a pricing foundation is problematic. We adopt the LMM framework as the central model due to its mathematical tractability and versatility for pricing a variety of instruments with the same analytical framework. In this subsection, we introduce an approach to finding an approximate dynamics of swap rates within the MLMM framework.

Consider the tenor structure \( \{T_0, T_1, \ldots, T_n\} \) with \( \delta = T_j - T_{j-1}, j = 1, 2, \ldots, n \). Define the \( N \)-year domestic forward swap rate, observed at time \( t \) and matured at time \( T_0 \) with \( 0 \leq t \leq T_0 \), as follows:

\[
S_d(t, T_0) = \sum_{j=1}^{n} w_j(t) L_d(t, T_{j-1}),
\]

where \( N = n\delta \) and \( w_j(t) \) is the weight of the \( j \)th LIBOR rate and defined by

\[
w_j(t) = \frac{\delta B_d(t, T_j)}{C(t; T_1, T_n)},
\]

and

\[
C(t; T_1, T_n) = \delta \sum_{k=1}^{n} B_d(t, T_k)
\]

We employ \( \tilde{W}(t) \) to denote an independent \( d \)-dimensional standard Brownian motion under an arbitrary measure.

The SMM is presented in Jamshidian (1997), under which the forward swap rate is a martingale and swaptions can be priced using the Black’s formula.

For simplicity, we set \( \delta \) as a constant. Its value may depend on the number of days in each accrual period, most often equal to three or six months.
denotes the present value of a basis point. According to (1), Equation (13) can be rewritten as

\[ S_d(t, T_0) = \frac{B_d(t, T_0) - B_d(t, T_n)}{C(t; T_1, T_n)}, \tag{16} \]

which can be regarded as a tradable asset expressed in units of \( C(t; T_1, T_n) \).

Assume that the market is arbitrage-free and complete and there exists an equivalent martingale measure with respect to the numéraire \( C(t; T_1, T_n) \) that is denoted by \( Q^C \). Under \( Q^C \), any tradable asset expressed in terms of \( C(t; T_1, T_n) \) must evolve as a martingale process, and thus the dynamics of \( S_d(t, T_0) \) should have the following form\(^{12}\):

\[ \frac{dS_d(t, T_0)}{S_d(t, T_0)} = \sigma_{S_d}(t, T_0) \cdot d\tilde{W}(t), \tag{17} \]

where \( \sigma_{S_d}(t, T_0) \), defined in (A1), denotes the volatility of a domestic forward swap rate within the MLMM framework. Because \( \sigma_{S_d}(t, T_0) \) is a stochastic process, \( S_d(T_0, T_0) \) is incomputable and its distribution is unknown. In Appendix A, we approximate \( \sigma_{S_d}(t, T_0) \) by a deterministic process \( \bar{\sigma}_{S_d}(t, T_0) \) defined by

\[ \bar{\sigma}_{S_d}(t, T_0) = \sum_{k=1}^{n} \bar{\zeta}_k^0(0)\gamma_d(t, t_{k-1}) + \sum_{k=1}^{n} \bar{\phi}_k^0(0)\bar{\sigma}_{p_d}(t, t_k) - \sum_{k=1}^{n} \bar{\zeta}_k^0(0)\bar{\sigma}_{p_d}(t, t_k), \tag{18} \]

where \( \bar{\phi}_k^0(0) \) and \( \bar{\zeta}_k^0(0) \) are defined in (A4) and (A5), respectively. In this way, \( S_d(T_0, T_0) \) can be computed by

\[ S_d(T_0, T_0) = S_d(0, T_0) \exp \left( -\frac{1}{2} \int_0^{T_0} \| \bar{\sigma}_{S_d}^0(t, T_0) \|^2 dt + \int_0^{T_0} \bar{\sigma}_{S_d}^0(t, T_0) \cdot d\tilde{W}(t) \right), \tag{19} \]

and \( \ln S_d(T_0, T_0) \) is normally distributed with mean and variance, defined, respectively, by

\[ M_d = \ln S_d(0, T_0) - \frac{1}{2} \int_0^{T_0} \| \bar{\sigma}_{S_d}^0(t, T_0) \|^2 dt, \]

\[ V_d^2 = \int_0^{T_0} \| \bar{\sigma}_{S_d}^0(t, T_0) \|^2 dt. \]

\(^{12}\)For details regarding this statement, refer to Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), and Jamshidian (1997).
2.3. The Generalized Lognormal Distribution

As we will see later, a CPSO can be regarded as an option on the spread between the domestic and the foreign swap rates, both of which are lognormally distributed. To price the CPSO, the distribution of the spread must be specified. To simplify the analysis, we reduce the problem to a simple structure. Assume that $S_d$ and $S_f$ are two lognormal random variables and $Y = S_d - S_f$. The goal is to compute the following expectation:

$$E[(Y - K)^+]$$

where $K$ is a constant and $(A)^+ = \max(A, 0)$. Although the distribution of $Y$ is unknown, its first three moments, $E[Y]$, $E[Y^2]$, and $E[Y^3]$, can be computed. In this section, we introduce a generalized family of lognormal distributions to approximate the distribution of $Y$.

The generalized family of lognormal distributions includes four types: regular, shifted, negative, and negative-shifted distributions. Their probability density functions are defined as follows.$^{13}$

**Definition 1:** The probability density functions of the generalized lognormal distributions are defined as follows:

(a) Regular lognormal distribution:

$$f(x) = \frac{1}{\beta x \sqrt{2\pi}} \exp\left(-\frac{1}{2\beta^2} (\log x - \alpha)^2\right), \quad x > 0.$$  \hspace{1cm} (21)

(b) Shifted lognormal distribution:

$$f(x) = \frac{1}{\beta(x - \gamma) \sqrt{2\pi}} \exp\left(-\frac{1}{2\beta^2} (\log(x - \gamma) - \alpha)^2\right), \quad x > \gamma.$$  \hspace{1cm} (22)

(c) Negative lognormal distribution:

$$f(x) = \frac{-1}{\beta x \sqrt{2\pi}} \exp\left(-\frac{1}{2\beta^2} (\log(-x) - \alpha)^2\right), \quad x < 0.$$  \hspace{1cm} (23)

$^{13}$For details regarding the generalized family of lognormal distributions, please refer to Johnson, Kotz, and Balakrishnan (1994) and Li, Deng, and Zhou (2008).
(d) Negative-shifted lognormal distribution:

\[ f(x) = \frac{1}{\beta(-x - \gamma)\sqrt{2\pi}} \exp\left(-\frac{1}{2\beta^2} (\log(-x - \gamma) - \alpha)^2\right), \quad x < -\gamma, \quad (24) \]

where \( \alpha, \beta, \) and \( \gamma \) denote, respectively, the scale, shape, and location parameters.

The relation among the above four distributions is given as follows: if a random variable \( X \) has a regular lognormal distribution, then the random variable \( X + \gamma \) has a shifted lognormal distribution; \( -X \) is a negative lognormal distribution; \( -(X + \gamma) \) is a negative-shifted lognormal distribution. We denote the first three moments of \( X \) by \( M_1(\alpha, \beta, \gamma) \), \( M_2(\alpha, \beta, \gamma) \), and \( M_3(\alpha, \beta, \gamma) \), which can be computed for each of the four types of distributions. The result is given in the following proposition.

**Proposition 5:** For each distribution, the first three moments in terms of parameters \( \alpha, \beta, \) and \( \gamma \) are presented as follows:

(a) If \( X \) has a regular lognormal distribution, then its first three moments are computed as follows:

\[ M_1(\alpha, \beta, \gamma) = \exp\left\{ \alpha + \frac{1}{2} \beta^2 \right\}, \]

\[ M_2(\alpha, \beta, \gamma) = \exp(2\alpha + 2\beta^2), \]

\[ M_3(\alpha, \beta, \gamma) = \exp\left\{ 3\alpha + \frac{9}{2} \beta^2 \right\}. \]

(b) If \( X \) has a shifted lognormal distribution, then its first three moments are computed as follows:

\[ M_1(\alpha, \beta, \gamma) = \gamma + \exp\left\{ \alpha + \frac{1}{2} \beta^2 \right\}, \]

\[ M_2(\alpha, \beta, \gamma) = \gamma^2 + 2\tau \exp\left\{ \alpha + \frac{1}{2} \beta^2 \right\} + \exp(2\alpha + 2\beta^2), \]

\[ M_3(\alpha, \beta, \gamma) = \gamma^3 + 3\gamma^2 \exp\left\{ \alpha + \frac{1}{2} \beta^2 \right\} + 3\gamma \exp(2\alpha + 2\beta^2) \]
+ \exp \left\{ 3\alpha + \frac{9}{2}\beta^2 \right\}.

(c) If $X$ has a negative lognormal distribution, then its first three moments are given by

\[ M_1(\alpha, \beta, \gamma) = -\exp\left\{ \alpha + \frac{1}{2}\beta^2 \right\}, \]
\[ M_2(\alpha, \beta, \gamma) = \exp(2\alpha + 2\beta^2), \]
\[ M_3(\alpha, \beta, \gamma) = -\exp\left\{ 3\alpha + \frac{9}{2}\beta^2 \right\}. \]

(d) If $X$ has a negative-shifted lognormal distribution, then its first three moments are given by

\[ M_1(\alpha, \beta, \gamma) = -\gamma - \exp\left\{ \alpha + \frac{1}{2}\beta^2 \right\}, \]
\[ M_2(\alpha, \beta, \gamma) = \gamma^2 + 2\gamma \exp\left\{ \alpha + \frac{1}{2}\beta^2 \right\} + \exp(2\alpha + 2\beta^2), \]
\[ M_3(\alpha, \beta, \gamma) = -\gamma^3 - 3\gamma^2 \exp\left\{ \alpha + \frac{1}{2}\beta^2 \right\} - 3\gamma \exp(2\alpha + 2\beta^2) \]
\[ - \exp\left\{ 3\alpha + \frac{9}{2}\beta^2 \right\}. \]

The parameters of a generalized lognormal random variable $X$ (which will be used to approximate $Y$) can be obtained by the moment matching method, and specifically, solving the following system of equations\(^{14}\):

\[ M_1(\alpha, \beta, \gamma) = E[Y] \]
\[ M_2(\alpha, \beta, \gamma) = E[Y^2] \implies \hat{\alpha}, \hat{\beta}, \hat{\gamma}. \quad (25) \]
\[ M_3(\alpha, \beta, \gamma) = E[Y^3] \]

The remaining problem is choosing an appropriate distribution $\hat{X}$ from the generalized family of lognormal distributions to approximate $Y$. The determinant

---

\(^{14}\)The analytical solution of this system of equation can be obtained in Chang, Chen, and Wu (2012).
criterion of the approximate distribution \( \hat{X} \) depends on both \( \hat{\gamma} \), computed from (25), and the skewness of \( Y \), which is defined as follows:

\[
\text{SK}_Y = \frac{E[Y - EY]^3}{(E[Y - EY]^2)^{1.5}}. \tag{26}
\]

The determinant procedure is specified as follows. If \( Y \) has a positive skewness (\( \text{SK}_Y > 0 \)), then the shifted lognormal random variable \( X \) is used to compute (25) and obtain \( \hat{\gamma} \). Next, if \( \hat{\gamma} \geq 0 \), then \( \hat{X} \) follows the regular lognormal distribution; otherwise, \( \hat{X} \) follows the shifted distribution. On the other hand, if \( Y \) has a negative skewness (\( \text{SK}_X < 0 \)), then the negative-shifted \( X \) is used to compute (25) and obtain \( \hat{\gamma} \). Next, if \( \hat{\gamma} \geq 0 \), then \( \hat{X} \) follows the negative lognormal distribution; otherwise, \( \hat{X} \) follows the negative-shifted distribution.

Once the approximate distribution \( \hat{X} \) is determined (one of the four types: (21)–(24)) and its parameters computed (from (25)), Equation (20) can be approximately computed by

\[
E[(Y - K)^+] \approx E[(\hat{X} - K)^+]. \tag{27}
\]

This approximate technique will be used to price CPSOs in Section 4.

3. PRICING AND HEDGING CPSs

In this section, we employ the MLMM to price and hedge two popular CPSs, both of which have domestic-currency notional principals. The first type is a plain-vanilla CPS, denoted by CPS\(_1\), which exchanges domestic-currency interest for foreign-currency interest. The second type is a variant CPS, denoted by CPS\(_2\), which swaps foreign-currency interest for another foreign-currency interest.

3.1. Pricing and Hedging CPS\(_1\)

A CPS\(_1\) is a contract that starts at time \( T_0 \) with reset dates \( T_0 < T_1 < \cdots < T_{n-1} \) and payment dates \( T_1 < T_2 < \cdots < T_n \). For simplicity, we define \( \delta = T_{j+1} - T_j \), \( j = 0, 1, \ldots, n - 1 \). During the tenor of a swap, notional principal, denoted by \( V_d \), is fixed and denominated in domestic currency. At each payment date \( T_j \) for \( j = 1, \ldots, n \), a domestic counterparty pays a foreign counterparty a domestic floating rate payment, \( \delta(L_d(T_{j-1}, T_j-1) + K) \), and receives from the foreign counterparty a foreign floating rate payment, \( \delta L_f(T_{j-1}, T_{j-1}) \). Both
Payments are made in domestic currency and $K$ denotes a spread in basis points.

To the domestic counterparty that pays the domestic floating rate payments and receives the foreign floating rate payment, the cash flow stream is given as follows:

At time $T_1$:
$$V_d \delta L_f(T_0, T_0) - V_d \delta [L_d(T_0, T_0) + K_1]$$

At time $T_2$:
$$V_d \delta L_f(T_1, T_1) - V_d \delta [L_d(T_1, T_1) + K_1]$$

... ... ...

At time $T_n$:
$$V_d \delta L_f(T_{n-1}, T_{n-1}) - V_d \delta [L_d(T_{n-1}, T_{n-1}) + K_1].$$

The pricing formula of the CPS1 is presented in the following theorem. The proof is given in Appendix B.

**Theorem 1**: The pricing formula of a CPS1. Within the MLMM framework, the price of a CPS1 at time $T_0$ is given by

$$CPS1(T_0) = V_d \delta \sum_{j=1}^{n} B_d(T_0, T_j) [L_f(T_0, T_j-1) - L_d(T_0, T_j-1) - K_1],$$

where

$$\tilde{\eta}^{T_0}_{j}(T_0, T_{j-1}) = \exp \left( \int_{T_0}^{T_{j-1}} \gamma_f(t, T_{j-1}) \cdot (\bar{\sigma}^{T_0}_{f}(t, T_j) - \bar{\sigma}^{T_0}_{d}(t, T_j) - \sigma_x(t)) dt \right).$$

Our CPS1 pricing formula within the MLMM framework has some practical advantages over the CPS1 pricing formulas in Turnbull (1993), Wei (1994), and Chang et al. (2002). First, the model parameters, including the initial LIBOR rates, exchange rate and volatilities, can be easily extracted from market data, thereby making our CPS1 pricing formula more feasible and tractable in practice. Second, the forward LIBOR rates within the MLMM framework are positive, thus avoiding pricing errors due to possible negative rates. In addition, most actively traded interest rate derivatives can be priced within the MLMM framework so that efficient risk management can be carried out consistently.

According to Equation (1), Equation (28) can be rewritten as follows:

$$V_d \sum_{j=1}^{n} \left\{ Q A_f(T_0, T_j) [B_f(T_0, T_{j-1}) - B_f(T_0, T_j)] - \delta K_1 B_d(T_0, T_j) \right\}$$
$$+ V_d (B_d (T_0, T_n) - 1),$$

(30)

where

$$QA_f (t, T_j) = \frac{B_d (t, T_j)}{B_f (t, T_j)} \hat{\eta}_f (T_0, T_j), \quad t \in [T_0, T_j].$$

$QA_f (T_0, T_j)$ denotes the quanto adjustment arising from a time-$T_j$ payoff that is measured in the $f$th country’s currency and made in domestic currency at time $T_0$. Therefore, $QA_f (T_0, T_j) B_f (T_0, T_{j-1})$ can be viewed as the price of a quanto bond (a foreign bond denominated in domestic currency).

CPS1s are developed to allow investors to capitalize on the expected differentials between the domestic and the foreign money market rates without incurring any direct exchange rate risk exposure. However, the exchange rate still affects the CPS1 price. Specifically, the CPS1 price is affected by both the exchange rate volatility and the correlation between the exchange rate and the foreign LIBOR rates. Both affect the CPS1 via the term $QA_f (T_0, \cdot)$.

Expression (30) provides an explicit construction of a hedging (replicating) portfolio for a CPS1. Because a CPS1 involves three different variates, including the domestic and the foreign interest rates and the exchange rate, it requires three financial instruments to construct a hedging portfolio, namely one domestic ZCB and two foreign ZCBs. To replicate a CPS1, an issuer must first borrow $V_d$ units of domestic currency and then use the fund to construct the following hedging portfolio, whose $j$th component, $j = 1, 2, \ldots, n$, is presented as follows:

$$P_j^{(1)} = \Delta^{(1)}_{1j} B_f (T_0, T_{j-1}) + \Delta^{(1)}_{2j} B_f (T_0, T_j) + \Delta^{(1)}_{3j} B_d (T_0, T_j),$$

where

$$\Delta^{(1)}_{1j} = + V_d QA_f (T_0, T_j),$$

$$\Delta^{(1)}_{2j} = - V_d QA_f (T_0, T_j),$$

$$\Delta^{(1)}_{3j} = \begin{cases} - V_d \delta K_1, & j = 1, 2, \ldots, n - 1, \\ + V_d (1 - \delta K_1), & j = n, \end{cases}$$

where “+” denotes a long position and “−” denotes a short position.
Pricing a newly issued CPS$_1$ is equivalent to computing a fair spread $\hat{K}_1$, which is given as follows:

$$\hat{K}_1 = \sum_{j=1}^{n} w_j(T_0)\bar{\eta}_f(T_0, T_{j-1}) - \sum_{j=1}^{n} w_j(T_0)L_d(T_0, T_{j-1}),$$  \hspace{1cm} (31)$$

where $w_j(T_0)$ is defined in (14). By setting the spread of the CPS$_1$ as $\hat{K}_1$, the CPS$_1$ becomes a fair contract. The fair spread $\hat{K}_1$ bears a resemblance to the difference between two weighted averages of the foreign and the domestic LIBOR rates. The spread $\hat{K}_1$ functions as a compensation for the counterparty that pays higher (domestic or foreign) interest rates and receives lower (foreign or domestic) interest rates. Besides, $\hat{K}_1$ is affected by the volatility of the foreign exchange rate and the correlation between the term structures of interest rates and the foreign exchange rate. Both affect $\hat{K}_1$ through the term $\bar{\eta}_f(T_0, T_{j-1})$.

### 3.2. Pricing and Hedging CPS$_2$

CPS$_2$ contracts, which exchange the $f$th country’s floating rate payments for the $g$th country’s floating rate payments, both denominated in domestic currency, are also popular in the over-the-counter market. CPS$_2$s allow investors to capitalize the expected differences between two foreign money market rates without incurring any direct exchange rate risk. International companies may use CPS$_2$s to transform their yields (assets) or costs (liabilities) from linking with the $f$th country’s interest rates to the $g$th country’s interest rates. Portfolio managers can employ CPS$_2$s to enhance current yields, and foreign currency borrowers can use them to lower their foreign interest rate costs, if they can take an accurate view on the relative movement of two foreign-currency interest rates.

Consider the same tenor structure and notations used in Section 3.1. To a domestic counterparty that pays the $f$th country’s floating rate payments and receives the $g$th country’s floating rate payments, the cash flow stream of a CPS$_2$ is given as follows:

At time $T_1$ : $V_d\delta L_g(T_0, T_0) - V_d\delta[L_f(T_0, T_0) + K_2]$

At time $T_2$ : $V_d\delta L_g(T_1, T_1) - V_d\delta[L_f(T_1, T_1) + K_2]$

$\vdots$

At time $T_n$ : $V_d\delta L_g(T_{n-1}, T_{n-1}) - V_d\delta[L_f(T_{n-1}, T_{n-1}) + K_2]$. 

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The pricing formula of a CPS$_2$ is presented in Theorem 2, whose proof is given in Appendix B.

**Theorem 2:** Pricing formula of a CPS$_2$. Within the MLMM framework, the price of a CPS$_2$ at time $T_0$ is given by

$$
\text{CPS}_2(T_0) = V_d \delta \sum_{j=1}^{n} B_d(T_0, T_j) \left\{ L_g(T_0, T_{j-1}) \bar{\eta}_g^{T_0}(T_0, T_{j-1}) \\
- L_f(T_0, T_{j-1}) \bar{\eta}_f^{T_0}(T_0, T_{j-1}) - K_2 \right\},
$$

where

$$
\bar{\eta}_g^{T_0}(T_0, T_{j-1}) = \exp \left( \int_{T_0}^{T_{j-1}} \gamma_g(t, T_{j-1}) \cdot (\tilde{\sigma}_g^{T_0}(t, T_j) - \tilde{\sigma}_B^{T_0}(t, T_j) - \sigma_X(t)) \, dt \right).
$$

Similar to (30), Equation (32) can be rewritten as follows:

$$
V_d \sum_{j=1}^{n} \left\{ Q_A_g(T_0, T_j) \left[ B_g(T_0, T_{j-1}) - B_g(T_0, T_j) \right] \\
- Q_A_f(T_0, T_j) \left[ B_f(T_0, T_{j-1}) - B_f(T_0, T_j) \right] - \delta K_2 B_d(T_0, T_j) \right\},
$$

where

$$
Q_A_g(t, T_j) = \frac{B_d(t, T_j)}{B_g(t, T_j)} \bar{\eta}_g^{T_0}(T_0, T_{j-1}), \quad t \in [T_0, T_j].
$$

$Q_A_g(T_0, \cdot)$ denotes the quanto adjustment that occurs when a payoff is measured in the $g$th country’s currency and made in domestic currency. Expression (34) provides an explicit way of constructing a hedging portfolio for a CPS$_2$. Because a CPS$_2$ involves five different terms (two foreign interest rates, one domestic interest rate and two exchange rates), it needs five financial instruments to construct a hedging portfolio (one domestic ZCB, two $f$th country’s ZCBs and two $g$th country’s ZCBs). To hedge a CPS$_2$, an issuer must construct the following hedging portfolio, whose $j$th component, $j = 1, 2, \ldots, n$, is represented as follows:

$$
P_j^{(2)} = \Delta_{1j}^{(2)} B_g(T_0, T_{j-1}) + \Delta_{2j}^{(2)} B_g(T_0, T_j) + \Delta_{3j}^{(2)} B_f(T_0, T_{j-1}) \\
+ \Delta_{4j}^{(2)} B_f(T_0, T_j) + \Delta_{5j}^{(2)} B_d(T_0, T_j),
$$
where

\[
\Delta_{1j}^{(2)} = +V_d QA_g(T_0, T_j), \\
\Delta_{2j}^{(2)} = -V_d QA_g(T_0, T_j), \\
\Delta_{3j}^{(2)} = -V_d QA_f(T_0, T_j), \\
\Delta_{4j}^{(2)} = +V_d QA_f(T_0, T_j), \\
\Delta_{5j}^{(2)} = -V_d \delta K_2.
\]

At an inception date \(T_0\), pricing a newly issued CPS\(_2\) is to compute a fair spread \(\hat{K}_2\) such that the CPS\(_2\) becomes a fair contract. The fair spread \(\hat{K}_2\) can be computed as follows:

\[
\hat{K}_2 = \sum_{j=1}^{n} w_j(T_0) L_g(T_0, T_{j-1}) \bar{\eta}_g^{T_0}(T_0, T_{j-1}) \\
- \sum_{j=1}^{n} w_j(T_0) L_f(T_0, T_{j-1}) \bar{\eta}_f^{T_0}(T_0, T_{j-1}),
\]

where \(w_j(T_0)\) is defined in (14). This bears a resemblance to the difference between two weighted averages of the \(g\)th country and the \(f\)th country LIBOR rates. The spread \(\hat{K}_2\) serves as a compensation for the counterparty that pays higher interest rates and receives lower interest rates. In addition, \(\hat{K}_2\) is affected by both the foreign exchange rate volatilities and the correlation between the term structures of interest rates and the foreign exchange rates, both of which are impounded in the terms \(\bar{\eta}_g^{T_0}(T_0, T_{j-1})\) and \(\bar{\eta}_f^{T_0}(T_0, T_{j-1})\).

4. PRICING AND HEDGING CPS\(_{SO}\)S

Consider the tenor structure of an underlying CPS defined in Section 3. A CPS\(_{SO}\) is an option on a CPS, and its cash flow at maturity date \(T_0\) (0 < \(T_0\) < \(T_1\)) is defined as follows:

\[
\text{CPSO}_k(T_0) = \text{Max}(\text{CPS}_k(T_0), 0), \quad k = 1 \text{ and } 2.
\]

The pricing formula of a CPS\(_{SO}\)\(_1\) with a zero spread (\(K_1 = 0\)) is examined in Schlögl (2002), whereas the pricing formula of a frequently traded CPS\(_{SO}\)\(_1\) with nonzero-spread (\(K_1 \geq 0\)) has (to the best of our knowledge) not yet been
studied. In this section we intend to derive, respectively, the pricing formulas of a CPSO$_1$ and a CPSO$_2$ with nonzero spreads.

4.1. Pricing and Hedging a CPSO$_1$ with Spread $K_1 \geq 0$

In this subsection, we derive the pricing formula of a CPSO$_1$ with a spread $K_1 \geq 0$. The resulting formula is more general and useful in practice than the case with $K_1 = 0$ derived in Schlögl (2002). According to (28), we can rewrite (36) as follows:

$$
\text{CPSO}_1(T_0) = V_d C(T_0; T_1, T_n) \text{Max}(Y_1 - K_1, 0),
$$

(37)

where

$$
Y_1 = S_f(T_0, T_0) - S_d(T_0, T_0),
$$

(38)

$$
C(t; T_1, T_n) = \delta \sum_{k=1}^{n} B_d(t, T_k),
$$

(39)

$$
S_f(t, T_0) = \sum_{j=1}^{n} w_j(t) L_f(t, T_{j-1}) \bar{\eta}_j^0(T_0, T_{j-1}),
$$

(40)

$$
S_d(t, T_0) = \sum_{j=1}^{n} w_j(t) L_d(t, T_{j-1}),
$$

(41)

$$
w_j(t) = \delta B_d(t, T_j)/C(t; T_1, T_n), \quad t \in [0, T_0].
$$

(42)

$S_d(\cdot, T_0)$ and $S_f(\cdot, T_0)$ represent a domestic and a quanto foreign forward swap rate, respectively, both of which are lognormally distributed according to the approximation technique presented in Section 2.2. In light of (37) and (38), a CPSO$_1$ can be regarded as a spread option on the difference between foreign and domestic swap rates. Based on the martingale pricing method and the approximation technique presented in Section 2.3, the pricing formula of a CPSO$_1$ can be derived under the martingale probability measure $Q^C$ with respect to the numéraire $C(t; T_1, T_n)$. The resulting formula is given in Theorem 3. Its proof is presented in Appendix C.
Theorem 3: Pricing formula of a CPSO with Spread \( K_1 \geq 0 \). Based on the martingale pricing method and the approximation technique presented in Section 2.3, the pricing formula of a CPSO with \( K_1 \geq 0 \) is computed within the MLMM framework as follows:

(a) If the approximate distribution for \( Y_1 \) is a regular lognormal distribution,

\[
\text{CPSO}_1(0) = V_0 C(0; T_1, T_n)((S_f(0, T_0) - S_d(0, T_0))N(a_{11}) - K_1 \times N(a_{12})),
\]

where \( a_{11} = \frac{\ln \left( \frac{S_f(0, T_0) - S_d(0, T_0)}{K_1} \right)}{\hat{\beta}_1} + \frac{1}{2} \hat{\beta}^2_1 \) and \( a_{12} = a_{11} - \hat{\beta}_1 \).

(b) If the approximate distribution for \( Y_1 \) is a shifted-regular lognormal distribution,

\[
\text{CPSO}_1(0) = V_0 C(0; T_1, T_n)((S_f(0, T_0) - S_d(0, T_0) - \hat{\gamma}_1)N(b_{11})
- (K_1 - \hat{\gamma}_1)N(b_{12})),
\]

where \( b_{11} = \frac{\ln \left( \frac{S_f(0, T_0) - S_d(0, T_0) - \hat{\gamma}_1}{K_1 - \hat{\gamma}_1} \right)}{\hat{\beta}_1} + \frac{1}{2} \hat{\beta}^2_1 \) and \( b_{12} = b_{11} - \hat{\beta}_1 \).

(c) If the approximate distribution for \( Y_1 \) is a negative lognormal distribution,

\[
\text{CPSO}_1(0) = V_0 C(0; T_1, T_n)((S_f(0, T_0) - S_d(0, T_0))N(c_{11}) - K_1 \times N(c_{12})),
\]

where \( c_{11} = -\frac{\ln \left( \frac{S_f(0, T_0) - S_d(0, T_0)}{K_1} \right)}{\hat{\beta}_1} + \frac{1}{2} \hat{\beta}^2_1 \) and \( c_{12} = c_{11} + \hat{\beta}_1 \).

(d) If the approximate distribution for \( Y_1 \) is a negative-shifted lognormal distribution,

\[
\text{CPSO}_1(0) = V_0 C(0; T_1, T_n)((S_f(0, T_0) - S_d(0, T_0) + \hat{\gamma}_1)N(d_{11})
- (K_1 + \hat{\gamma}_1)N(d_{12})),
\]

where \( d_{11} = -\frac{\ln \left( \frac{S_f(0, T_0) - S_d(0, T_0) + \hat{\gamma}_1}{K_1 + \hat{\gamma}_1} \right)}{\hat{\beta}_1} + \frac{1}{2} \hat{\beta}^2_1 \) and \( d_{12} = d_{11} + \hat{\beta}_1 \).
Using the system of equations in (25), the parameters, \( \hat{\alpha}_1, \hat{\beta}_1, \) and \( \hat{\gamma}_1, \) can be computed in terms of \( E^{Q_t}[Y_1], E^{Q_t}[Y_2], \) and \( E^{Q_t}[Y_3], \) which are defined in (C4)–(C6), respectively, in Appendix C.

The CPSO\(_1\) pricing formulas shown in (43)–(46) resemble the swaption pricing formula presented in Jamshidian (1997), except that the underlying swap rate is replaced by a difference between foreign and domestic swap rates. Compared to the formula \((K_1 = 0)\) presented in Schlögl (2002), our pricing formulas are more capable of pricing a CPSO\(_1\) with spread \( K_1 > 0, \) which is more frequently traded than that with zero spread \((K_1 = 0)\) in the marketplace. In addition, the parameters required to compute the CPSO\(_1\) price can be easily extracted from market data, thereby making the pricing formulas more feasible and tractable for market practitioners.

Because Equations (43)–(46) are derived under the MLMM framework, they provide us a hedging strategy that is consistent with caps, floors, and other exotic interest rate derivatives. For further examination regarding an explicit hedging strategy, we take (43) as a demonstration and other related cases can be performed accordingly. Equation (43) can be rewritten as follows:

\[
\text{CPSO}_1(0) = V_d \sum_{j=1}^{n} \left\{ QA_f(0, T_j)N(a_{11})[B_f(0, T_{j-1}) - B_f(0, T_j)] \right. \\
- \left. \delta K_1 N(a_{12})B_d(0, T_j) \right\} + V_d N(a_{11})(B_d(0, T_n) - B_d(0, T_0)). \tag{47}
\]

If a financial institution issues a CPSO\(_1\), then Equation (47) reveals a clue for constructing a hedging portfolio composed of various long and short positions in \( B_d(0, \cdot) \) and \( B_f(0, \cdot), \) whose \( j \)th component, \( j = 1, 2, \ldots, n, \) is established by

\[
P^{(3)}_j = \Delta^{(3)}_{1j}B_f(0, T_{j-1}) + \Delta^{(3)}_{2j}B_f(0, T_j) + \Delta^{(3)}_{3j}B_d(0, T_j) + \Delta^{(3)}_{4j}B_d(0, T_{j-1}),
\]

where

\[
\Delta^{(3)}_{1j} = +V_dQA_f(0, T_j)N(a_{11}),
\]

\[
\Delta^{(3)}_{2j} = -V_dQA_f(0, T_j)N(a_{11}),
\]

\[
\Delta^{(3)}_{3j} = \begin{cases} 
-V_d\delta K_1 N(a_{12}), & j = 1, 2, \ldots, n-1, \\
+V_dN(a_{11}) - V_d\delta K_1 N(a_{12}), & j = n,
\end{cases}
\]

\[
\Delta^{(3)}_{4j} = \begin{cases} 
-V_dN(a_{11}), & j = 1, \\
0, & j = 2, 3, \ldots, n.
\end{cases}
\]
4.2. Pricing and Hedging a CPSO with Spread $K_2 \geq 0$

In this subsection, we present the pricing formula of a CPSO with a spread $K_2 \geq 0$. According to (32) and (36), we may rewrite the final payoff of a CPSO as follows:

$$\text{CPSO}_2(T_0) = V_d C(T_0; T_1, T_n) \text{Max}(Y_2 - K_2, 0),$$

where

$$Y_2 = S_g(T_0, T_0) - S_f(T_0, T_0),$$

$$S_g(t, T_0) = \sum_{j=1}^{n} w_j(t)(L_g(t, T_j - 1)\bar{\eta}_g(T_0, T_j - 1)), \quad t \in [0, T_0].$$

$S_g(\cdot, T_0)$ and $S_f(\cdot, T_0)$ represent, respectively, the $g$th country’s and the $f$th country’s quanto forward swap rates, both of which are lognormally distributed according to the approximation technique described in Section 2.2. Therefore, a CPSO can be considered a spread option on the difference between two quanto foreign swap rates. Based on the martingale pricing method and the approximation technique presented in Section 2.3, the pricing formula of a CPSO can be derived, and is given in Theorem 4. Its proof is omitted for the sake of brevity.

**Theorem 4:** Pricing formula of a CPSO with spread $K_2 \geq 0$. Based on the martingale pricing method and the approximation technique presented in Section 2.3, the pricing formula of a CPSO with $K_2 \geq 0$ is computed within the MLMM framework as follows:

(a) If the approximate distribution for $Y_2$ is a regular lognormal distribution,

$$\text{CPSO}_2(0) = V_d C(0; T_1, T_n)((S_g(0, T_0) - S_f(0, T_0))N(a_{21}) - K_2 \times N(a_{22})),$$

where $a_{21} = \frac{\ln \left( \frac{S_g(0, T_0) - S_f(0, T_0)}{K_2} \right) + \frac{1}{2} \hat{\beta}_2^2}{\hat{\beta}_2}$ and $a_{22} = a_{21} - \hat{\beta}_2$. 
(b) If the approximate distribution for $Y_2$ is a shifted-regular lognormal distribution,

$$\text{CPSO}_2(0) = V_d C(0; T_1, T_n)((S_g(0, T_0) - S_f(0, T_0) - \hat{\gamma}_2)N(b_{21})$$

$$- (K_2 - \hat{\gamma}_2)N(b_{22})),$$

where $b_{21} = \frac{\ln \left( \frac{S_g(0, T_0) - S_f(0, T_0) - \hat{\gamma}_2}{K_2 - \hat{\gamma}_2} \right) + \frac{1}{2} \beta_2^2}{\beta_2}$ and $b_{22} = b_{21} - \hat{\beta}_2$.

(c) If the approximate distribution for $Y_2$ is a negative lognormal distribution,

$$\text{CPSO}_2(0) = V_d C(0; T_1, T_n)((S_g(0, T_0) - S_f(0, T_0))N(c_{21}) - K_2 \times N(c_{22})),$$

where $c_{21} = -\frac{\ln \left( \frac{S_g(0, T_0) - S_f(0, T_0)}{K_2} \right) + \frac{1}{2} \beta_2^2}{\hat{\beta}_2}$ and $c_{22} = c_{21} + \hat{\beta}_2$.

(d) If the approximate distribution for $Y_2$ is a negative-shifted lognormal distribution,

$$\text{CPSO}_2(0) = V_d C(0; T_1, T_n)((S_g(0, T_0) - S_f(0, T_0) + \hat{\gamma}_2)N(d_{21})$$

$$- (K_2 + \hat{\gamma}_2)N(d_{22})),$$

where $d_{21} = -\frac{\ln \left( \frac{S_g(0, T_0) - S_f(0, T_0) + \hat{\gamma}_2}{K_2 + \hat{\gamma}_2} \right) + \frac{1}{2} \beta_2^2}{\hat{\beta}_2}$ and $d_{22} = d_{21} + \hat{\beta}_2$.

Using the system of equations in (25), the parameters, $\hat{\alpha}_2$, $\hat{\beta}_2$, and $\hat{\gamma}_2$, can be computed in terms of $E^{Q^C}[Y_2]$, $E^{Q^C}[Y_2^2]$, and $E^{Q^C}[Y_2^3]$, which are presented in (C15)–(C17), respectively.

The parameters defined in Equations (51)–(54) can be easily extracted from market data. Due to the versatility of the MLMM, the CPSO_2 pricing formulas lead us to a hedging strategy that is consistent with other related exotic interest rate derivatives. We take (52) as a demonstration and other related cases can be inferred accordingly. Equation (52) can be rewritten as follows:
Currency-Protected Swaps and Swaptions

\[ \text{CPSO}_2(0) = V_d \sum_{j=1}^{n} \left\{ Q A_g(0, T_j) N(b_{21}) \left[ B_g(0, T_{j-1}) - B_g(0, T_j) \right] - Q A_f(0, T_j) N(b_{21}) \left[ B_f(0, T_{j-1}) - B_f(0, T_j) \right] - \left[ \delta \hat{\gamma}_2 N(b_{21}) + \delta (K_2 - \hat{\gamma}_2) N(b_{22}) \right] B_d(0, T_j) \right\} \]

(55)

If a financial institution issues a CPSO\(_2\), then the below hedging portfolio can be constructed, and its interpretation is similar to (47):

\[ P^{(4)}_j = \Delta^{(4)}_{1j} B_g(0, T_{j-1}) + \Delta^{(4)}_{2j} B_g(0, T_j) + \Delta^{(4)}_{3j} B_f(0, T_{j-1}) + \Delta^{(4)}_{4j} B_f(0, T_j) + \Delta^{(4)}_{5j} B_d(0, T_j), \]

where

\[ \Delta^{(4)}_{1j} = +V_d \ Q A_g(0, T_j) N(b_{21}), \]
\[ \Delta^{(4)}_{2j} = -V_d \ Q A_g(0, T_j) N(b_{21}), \]
\[ \Delta^{(4)}_{3j} = -V_d \ Q A_f(0, T_j) N(b_{21}), \]
\[ \Delta^{(4)}_{4j} = +V_d \ Q A_f(0, T_j) N(b_{21}), \]
\[ \Delta^{(4)}_{5j} = -V_d \left[ \delta \hat{\gamma}_2 N(b_{21}) + \delta (K_2 - \hat{\gamma}_2) N(b_{22}) \right]. \]

5. NUMERICAL STUDIES

In this section, we provide numerical examples for analyzing the accuracy of the pricing formulas developed here by comparing the results to those of a Monte Carlo (MC) simulation. We only present the results for Theorems 2 and 4, while those for Theorems 1 and 3 are similar, and are thus omitted to conserve space. The results show that our pricing formulas are robustly and sufficiently accurate. We introduce the implementation of the numerical studies in detail as follows.\(^{15}\)

First, we consider CPS\(_2\)s that pay Japanese LIBOR rates plus a spread and receive U.K. LIBOR rates, both denominated in U.S. dollar. The CPS\(_2\)s considered have maturities of three, five, and seven years, and the spread is 150 basis points. Second, we consider CPSO\(_2\)s on the five-year CPS\(_2\) that pay Japanese LIBOR rates plus a spread and receive U.K. LIBOR rates, both

\(^{15}\)For the parameter calibration of the MLMM, please refer to Wu and Chen (2007a, 2007b).
TABLE I
Numerical Examples of CPS$_2$ in Basis Points

<table>
<thead>
<tr>
<th>Date</th>
<th>Three-year CPS$_2$</th>
<th>Five-year CPS$_2$</th>
<th>Seven-year CPS$_2$</th>
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<tr>
<td></td>
<td>Theorem 2</td>
<td>MC</td>
<td>Theorem 2</td>
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<td>144.1</td>
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The three-, five-, and seven-year CPS$_2$s that pay Japanese LIBOR rates plus 150 basis points and receive U.K. LIBOR rates, both denominated in U.S. dollars, are seasonally priced based on the market data in 2007 and 2008. The notional value is assumed to be $1. The simulation is implemented based on 30,000 paths. MC stands for the result of the Monte Carlo simulation. The standard error of each simulation is about 2 basis points, on average.

denominated in U.S. dollar. The maturity periods of CPSO$_2$s are one, three, and five years, and the spreads of their underlying CPS$_2$ are 50, 100, 150, 200, and 250 basis points.

To examine the accuracy and robustness of the derived formula under different market scenarios, both CPS$_2$s and CPSO$_2$s are priced using market data chosen quarterly in 2007 and 2008. The notional principal for each case is assumed to be $1 and each simulation is implemented based on 30,000 paths. The results are listed in Tables I and II, and show that the approximate formulas are sufficiently accurate as compared to the MC simulation and robust under different market scenarios over the past two years.

6. CONCLUSION

Within the framework of a multifactor MLMM, we derive the approximate pricing formulas for CPS$_1$s, CPS$_2$s, CPSO$_1$s, and CPSO$_2$s with nonzero spreads. As compared with traditional interest rate models, the LIBOR rates specified in the MLMM are market-observable and the model parameters can be easily extracted from market data. Therefore, our pricing models are feasible and tractable for practical implementation.

Without our pricing formulas, CPSs and CPSOs are generally computed in practice using inefficient and time-consuming MC simulation. As shown in Section 5, our pricing formulas yield prices close to those computed in MC simulations, but in a fraction of the time. Our formulas provide market

16The market data used for numerical studies are available upon request from the authors.


TABLE II
Numerical Examples of CPSO<sub>2</sub>s in Basis Points

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<tr>
<th>Date</th>
<th>Spread</th>
<th>One-year CPSO&lt;sub&gt;2&lt;/sub&gt;</th>
<th>Three-year CPSO&lt;sub&gt;2&lt;/sub&gt;</th>
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<td>MC</td>
<td>Theorem 4</td>
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The one-, two-, and three-year CPSO<sub>2</sub>s on a CPS<sub>2</sub> that pay Japanese LIBOR rates plus a spread and receive U.K. LIBOR rates, both denominated in U.S. dollars, are quarterly priced based on the market data in 2007 and 2008. The spreads considered are 50, 100, 150, 200, and 250 basis points. The notional value is assumed to be $1. The simulation is implemented based on 30,000 paths. MC stands for the result of the Monte Carlo simulation. The standard error of each simulation is about 2 basis points, on average.
practitioners with a new, efficient and time-saving approach for offering almost instantly quoted prices to clients and the daily marking-to-market trading books, and facilitating efficient risk management of trading positions. Given these advantages, the presented formulas are worth recommending to market practitioners.

APPENDIX A: DERIVATION OF FORWARD SWAP RATE VOLATILITIES

This appendix intends to derive forward swap rate volatilities, $\bar{\sigma}_{Sd}^{0}(t, T_{0})$, $\bar{\sigma}_{Sf}^{0}(t, T_{0})$ and $\bar{\sigma}_{Sg}^{0}(t, T_{0})$, within the MLMM framework. We first derive the domestic swap rate volatility, $\bar{\sigma}_{Sd}^{0}(t, T_{0})$. Taking logarithm to (13), we have

$$\ln(S_{d}(t, T_{0})) = \ln\left(\sum_{i=1}^{n} B_{d}(t, T_{i})L_{d}(t, T_{i-1})\right) - \ln\left(\sum_{i=1}^{n} B_{d}(t, T_{i})\right),$$

and

$$\frac{\partial \ln(S_{d}(t, T_{0}))}{\partial B_{d}(t, T_{k})} = \frac{L_{d}(t, T_{k-1})}{\sum_{i=1}^{n} B_{d}(t, T_{i})L_{d}(t, T_{i-1})} - \frac{1}{\sum_{i=1}^{n} B_{d}(t, T_{i})},$$

$$\frac{\partial \ln(S_{d}(t))}{\partial L_{d}(t, T_{k-1})} = \frac{B_{d}(t, T_{k})}{\sum_{i=1}^{n} B_{d}(t, T_{i})L_{d}(t, T_{i-1})}, \text{ for } k = 1, 2, \ldots, n.$$

By employing Itô’s lemma, $\sigma_{Sd}(t, T_{0})$ is derived as follows:

$$\sigma_{Sd}(t, T_{0}) = \sum_{k=1}^{n} \left[ \frac{\partial \ln S_{d}(t, T_{0})}{\partial B_{d}(t, T_{k})} (-\bar{\sigma}_{Bd}^{0}(t, T_{k})B_{d}(t, T_{k})) + \frac{\partial \ln S_{d}(t, T_{0})}{\partial L_{d}(t, T_{k-1})} \gamma_{d}(t, T_{k-1})L_{d}(t, T_{k-1}) \right]$$

$$= \sum_{k=1}^{n} \zeta_{k}(t)\gamma_{d}(t, T_{k-1}) + \sum_{k=1}^{n} \phi_{k}(t)\bar{\sigma}_{Bd}^{0}(t, T_{k}) - \sum_{k=1}^{n} \zeta_{k}(t)\bar{\sigma}_{Bd}^{0}(t, T_{k}), \quad (A1)$$

where

$$\phi_{k}(t) = \frac{B_{d}(t, T_{k})}{\sum_{i=1}^{n} B_{d}(t, T_{i})}, \quad (A2)$$
\[ \zeta_k(t) = \frac{B_d(t, T_k)L_d(t, T_{k-1})}{\sum_{i=1}^{n} B_d(t, T_i)L_d(t, T_{i-1})}. \]  \hspace{1cm} (A3)

Both \( \phi_k(t) \) and \( \zeta_k(t) \) look like weights, and thus \( \sigma_{Sd}(t, T_0) \) can be regarded as a composite of three weighted averages. Because the weights \( \phi_k(t) \) and \( \zeta_k(t) \), for \( k = 1, 2, \ldots, n \), are stochastic, so is \( \sigma_{Sd}(t, T_0) \). Although the variability of \( \phi_k(t) \) and \( \zeta_k(t) \) are small (to be shown later), we can freeze the calendar time of both \( B_d(t, \cdot) \) and \( L_d(t, \cdot) \) in (A2) and (A3) at its initial time 0, and the resulting processes become deterministic and are defined by

\[ \bar{\phi}_0^0(0) = \frac{B_d(0, T_k)}{\sum_{i=1}^{n} B_d(0, T_i)} \]  \hspace{1cm} (A4)

\[ \bar{\zeta}_0^0(0) = \frac{B_d(0, T_k)L_d(0, T_{k-1})}{\sum_{i=1}^{n} B_d(0, T_i)L_d(0, T_{i-1})} \]  \hspace{1cm} (A5)

By replacing \( \phi_k(t) \) and \( \zeta_k(t) \), respectively, by \( \bar{\phi}_0^0(0) \) and \( \bar{\zeta}_0^0(0) \), \( \bar{\sigma}_{Sd}^0(t, T_0) \) becomes deterministic and given by

\[ \bar{\sigma}_{Sd}^0(t, T_0) = \sum_{k=1}^{n} \bar{\zeta}_0^0(0)\gamma_d(t, T_{k-1}) + \sum_{k=1}^{n} \bar{\phi}_0^0(0)\bar{\sigma}_{B_d}^0(t, T_k) - \sum_{k=1}^{n} \bar{\zeta}_0^0(0)\bar{\sigma}_{B_d}^0(t, T_k). \]  \hspace{1cm} (A6)

This approximation is widely employed in practice to price swap-type instruments within the MLMM framework and examined to be robustly accurate in the numerical examples given in Section 5. For the rest of this appendix, we provide the main theoretical foundation to show that the variability of the weights, \( \phi_k(t) \) and \( \zeta_k(t) \), is small, and thus it is reasonable to freeze these processes at time 0. For the sake of brevity, we only show the weight \( \zeta_k(t) \) for a demonstration. Rewrite \( \zeta_k(t) \) as follows:

\[ \zeta_k(t) = \frac{Y(t, T_k)}{\sum_{i=1}^{n} Y(t, T_i)}, \]

where \( Y(t, T_i) = B_d(t, T_i)L_d(t, T_{i-1}) \) for \( i = 1, 2, \ldots, n \).
Based on Itô’s lemma, the dynamics of $\zeta_k(t)$ can be derived and its volatility term, $\sigma_{\zeta_k}(t)$, is presented as follows:

$$
\sigma_{\zeta_k}(t) = \sum_{i=1}^{n} \left[ \frac{1}{\sum_{j=1}^{n} Y(t, T_j)} \left( \sigma_{Bd}(t, T_j) - \gamma_d(t, T_{i-1}) \right) \right] 
+ \frac{1}{Y(t, T_k)} \left( \gamma_d(t, T_{k-1}) - \sigma_{Bd}(t, T_k) \right) 
= \frac{1}{Y(t, T_k)} \left[ \sum_{i=1}^{n} \zeta_i(t) \left( \sigma_{Bd}(t, T_i) - \gamma_d(t, T_{i-1}) \right) + \left( \gamma_d(t, T_{k-1}) - \sigma_{Bd}(t, T_k) \right) \right].
$$

Because $Y(t, T_i) \approx Y(t, T_k)$, $\sigma_{\zeta_k}(t)$ can be approximately rewritten as follows:

$$
\sigma_{\zeta_k}(t) = \frac{1}{Y(t, T_k)}
\times \left[ \sum_{i=1}^{n} \zeta_i(t) \sigma_{Bd}(t, T_i) - \sigma_{Bd}(t, T_k) + \gamma_d(t, T_{k-1}) - \sum_{i=1}^{n} \zeta_i(t) \gamma_d(t, T_{i-1}) \right].
$$

(A7) (A8)

Because the plus and minus terms in (A7) and (A8) are roughly equal, the volatility of $\zeta_k(t)$ tends to be small. Therefore, this fact justifies that $\tilde{\zeta}_k^0(0)$ is a good approximation to $\zeta_k(t)$. Similarly, we can also approximate $\phi_k(t)$ by $\tilde{\phi}_k^0(0)$.

Similarly, the swap volatilities of the $f$th and $g$th country’s swap rates, $\tilde{\sigma}_{S_f}^0(t, T_0)$ and $\tilde{\sigma}_{S_g}^0(t, T_0)$, can be derived as follows:

$$
\tilde{\sigma}_{S_f}^0(t, T_0) = \sum_{k=1}^{n} \tilde{\theta}_k^0(0) \gamma_f(t, T_{k-1}) + \sum_{k=1}^{n} \tilde{\phi}_k^0(0) \tilde{\sigma}_{Bd}^0(t, T_k) - \sum_{k=1}^{n} \tilde{\theta}_k^0(0) \tilde{\sigma}_{Bd}^0(t, T_k),
$$

(A9)

$$
\tilde{\sigma}_{S_g}^0(t, T_0) = \sum_{k=1}^{n} \tilde{\psi}_k^0(0) \gamma_g(t, T_{k-1}) + \sum_{k=1}^{n} \tilde{\phi}_k^0(0) \tilde{\sigma}_{Bd}^0(t, T_k) - \sum_{k=1}^{n} \tilde{\psi}_k^0(0) \tilde{\sigma}_{Bd}^0(t, T_k).
$$

(A10)
where

\[ \tilde{\eta}_0^f(T_0, T_{j-1}) = \sum_{i=1}^{n} B_d(0, T_i)L_f(0, T_{i-1}) \tilde{\eta}_0^0(T_0, T_{i-1}) \]  
\[ \tilde{\eta}_0^g(T_0, T_{j-1}) = \sum_{i=1}^{n} B_d(0, T_i)L_g(0, T_{i-1}) \tilde{\eta}_0^0(T_0, T_{i-1}) \]  

(A11)

where \( \tilde{\eta}_0^0(T_0, T_{i-1}) \) and \( \tilde{\eta}_0^0(T_0, T_{i-1}) \) are defined in (B3).

**APPENDIX B: PROOF OF THEOREMS 1 AND 2**

Based on the changing-measure mechanism in Proposition 4, the dynamics of \( L_d(T_j, T_{j-1}) \) and \( L_k(T_j, T_{j-1}) \), \( k \in \{f, g\} \), under \( Q^T \) are given as follows:

\[
\frac{dL_d(t, T_{j-1})}{L_d(t, T_{j-1})} = \gamma_d(t, T_{j-1}) \cdot d\tilde{W}(t), \tag{B1}
\]

\[
\frac{dL_k(t, T_{j-1})}{L_k(t, T_{j-1})} = \gamma_k(t, T_{j-1}) \cdot \left( \tilde{\sigma}_f(t, T_j) - \tilde{\sigma}_d(t, T_j) - \sigma_k(t) \right) dt + \gamma_k(t, T_{j-1}) \cdot d\tilde{W}(t), \tag{B2}
\]

and

\[
L_d(T_{j-1}, T_{j-1}) = L_d(T_0, T_{j-1}) \exp \left( -\frac{1}{2} \int_{T_0}^{T_{j-1}} \| \gamma_d(t, T_{j-1}) \|^2 dt + \int_{T_0}^{T_{j-1}} \gamma_d(t, T_{j-1}) \cdot d\tilde{W}(t) \right),
\]

\[
L_k(T_{j-1}, T_{j-1}) = L_k(T_0, T_{j-1}) \tilde{\eta}_0^{T_0}(T_0, T_{j-1}) \exp \left( -\frac{1}{2} \int_{T_0}^{T_{j-1}} \| \gamma_d(t, T_{j-1}) \|^2 dt + \int_{T_0}^{T_{j-1}} \gamma_d(t, T_{j-1}) \cdot d\tilde{W}(t) \right),
\]

where

\[
\tilde{\eta}_0^{T_0}(T_0, T_{j-1}) = \exp \left( \int_{T_0}^{T_{j-1}} \gamma_k(t, T_{j-1}) \cdot \left( \tilde{\sigma}_f(t, T_j) - \tilde{\sigma}_d(t, T_j) - \sigma_k(t) \right) dt \right). \tag{B3}
\]
Based on the martingale pricing method, the price of a CPS1 can be computed as follows:

\[
CPS_1(T_0) = \delta \sum_{j=1}^{n} B_d(T_0, T_j) E^{Q^T_0} \left[ L_f(T_{j-1}, T_{j-1}) - L_d(T_{j-1}, T_{j-1}) - K_1 | F_{T_0} \right]
\]

\[
= \delta \sum_{j=1}^{n} B_d(T_0, T_j) \left\{ E^{Q^T_0} [L_f(T_{j-1}, T_{j-1}) | F_{T_0}] - L_d(T_0, T_{j-1}) - K_1 \right\}
\]

\[
= \delta \sum_{j=1}^{n} B_d(T_0, T_j) \{L_f(T_0, T_{j-1}) \bar{\eta}_f^{T_0}(T_0, T_{j-1}) - L_d(T_0, T_{j-1}) - K_1\}.
\]

Similarly, the price of a CPS2 can be computed as follows:

\[
CPS_2(T_0) = \delta \sum_{j=1}^{n} B_d(T_0, T_j) \{L_g(T_0, T_{j-1}) \bar{\eta}_g^{T_0}(T_0, T_{j-1})
\]

\[
- L_f(T_0, T_{j-1}) \bar{\eta}_f^{T_0}(T_0, T_{j-1}) - K_2\}.
\]

**APPENDIX C: PROOF OF THEOREM 3**

Based on the martingale pricing method, the price of a CPSO1 at time 0 is computed as follows:

\[
CPSO_1(0)
\]

\[
= B_d(0, T_0) E^{Q^{0_0}} \left\{ \text{Max}(CPS_1(T_0), 0) | F_0 \right\}
\]

\[
= V_d B_d(0, T_0) E^{Q^{0_0}} \left\{ C(T_0; T_1, T_n) \text{Max}(S_f(T_0, T_0) - S_d(T_0, T_0) - K_1, 0) | F_0 \right\}
\]

\[
= V_d C(0; T_1, T_n) E^{Q^{0_0}} \left\{ \frac{d Q^C}{d Q^{0_0}} \text{Max}(S_f(T_0, T_0) - S_d(T_0, T_0) - K_1, 0) | F_0 \right\}
\]

\[
= V_d C(0; T_1, T_n) E^{Q^C} \left\{ \text{Max}(S_f(T_0, T_0) - S_d(T_0, T_0) - K_1, 0) | F_0 \right\}.
\]

(C1)

where

\[
\frac{d Q^C}{d Q^{0_0}} = \frac{C(T_0; T_1, T_n)/C(0; T_1, T_n)}{B_d(T_0, T_0)/B_d(0, T_0)}
\]
denotes the Randon–Nikodym derivative that defines a martingale probability measure with respect to the numéraire \( C(t; T_1, T_n) \) which is given by

\[
\mathcal{Q}^C(\mathcal{A}) = \int_{\omega \in \mathcal{A}} d\mathcal{Q}^C_0 d\mathcal{Q}^C_0, \quad \forall \mathcal{A} \in \mathcal{F}.
\]

According to (40) and (41), both \( S_d(t, T_0) \) and \( S_f(t, T_0) \) can be regarded as tradable assets expressed in \( C(t; T_1, T_n) \) units, and thus must evolve as martingale processes under the measure \( \mathcal{Q}^C \). Based on the approximate technique presented in Section 2.2, the dynamics of \( S_d(t, T_0) \) and \( S_f(t, T_0) \) should have the following form under the measure \( \mathcal{Q}^C \):

\[
\frac{dS_k(t, T_0)}{S_k(t, T_0)} = \bar{\sigma}^0_{Sk}(t) \cdot d\tilde{W}(t),
\]

and

\[
S_k(T_0, T_0) = S_k(0, T_0) \exp \left( -\frac{1}{2} \int_0^{T_0} \|\bar{\sigma}^0_{Sk}(t)\|^2 dt + \int_0^{T_0} \bar{\sigma}^0_{Sk}(t) \cdot d\tilde{W}(t) \right),
\]

\[k \in \{d, f\},\quad (C2)\]

where \( \bar{\sigma}^0_{Sd}(t) \) and \( \bar{\sigma}^0_{Sf}(t) \) are defined in (A6) and (A9), respectively. Therefore, \( S_k(T_0, T_0) \) has a lognormal distribution with \( \ln S_k(T_0, T_0) \sim N(M_k, V_k^2) \), where

\[
M_k = \ln S_k(0, T_0) - \frac{1}{2} \int_0^{T_0} \|\bar{\sigma}^0_{Sk}(t)\|^2 dt, \quad (C3a)
\]

\[
V_k^2 = \int_0^{T_0} \|\bar{\sigma}^0_{Sk}(t)\|^2 dt. \quad (C3b)
\]

To compute (C1), we can use the approximation technique presented in Section 2.3. Let \( Y_1 = S_f(T_0, T_0) - S_d(T_0, T_0) \), and its first three moments are computed as follows:

\[
\mathbb{E}^{\mathcal{Q}^C}[Y_1] = S_f(0, T_0) - S_d(0, T_0), \quad (C4)
\]

\[
\mathbb{E}^{\mathcal{Q}^C}[Y_1^2] = S_f(0, T_0)^2 \exp \left( \int_0^{T_0} \|\bar{\sigma}^0_{Sf}(t)\|^2 dt \right) + S_d(0, T_0)^2 \exp \left( \int_0^{T_0} \|\bar{\sigma}^0_{Sd}(t)\|^2 dt \right) - 2S_f(0, T_0)S_d(0, T_0) \exp \left( \int_0^{T_0} \bar{\sigma}^0_{Sd}(t) \cdot \bar{\sigma}^0_{Sf}(t) dt \right), \quad (C5)
\]
\[ E^{QC}[Y_1^3] = S_f(0, T_0)^3 \exp \left( 3 \int_0^{T_0} \| \bar{\sigma}_S^0(t) \|^2 dt \right) \]
\[ - S_d(0, T_0)^3 \exp \left( 3 \int_0^{T_0} \| \bar{\sigma}_d^0(t) \|^2 dt \right) \]
\[ - 3S_f(0, T_0)^2 S_d(0, T_0) \exp \left( \int_0^{T_0} \| \bar{\sigma}_S^0(t) \|^2 + 2\bar{\sigma}_d^0(t) \cdot \bar{\sigma}_S^0(t) dt \right) \]
\[ + 3S_f(0, T_0) S_d(0, T_0)^2 \exp \left( \int_0^{T_0} \| \bar{\sigma}_d^0(t) \|^2 + 2\bar{\sigma}_d^0(t) \cdot \bar{\sigma}_S^0(t) dt \right) \].

(C6)

Solving the system of equations in (25) for \( \hat{\alpha}_1, \hat{\beta}_1, \) and \( \gamma_1 \) and computing the skewness of \( Y_1 \) in (26), we can find a generalized lognormal distribution \( \hat{X}_1 \) to approximate \( Y_1 \), and the price of a CPSO1 can be computed for each type of \( \hat{X}_1 \) as follows:

\[ \text{CPSO}_1(0) = V_d C(0; T_1, T_n) E^{QC} \{ \max(\hat{X}_1 - K_1, 0) \} | \mathcal{F}_0 \]  

(C7)

\[ = V_d C(0; T_1, T_n) \left( E^{QC}[\hat{X}_1 I_{\{\hat{X}_1 \geq K_1\}}] - K_1 E^{QC}[I_{\{\hat{X}_1 \geq K_1\}}] \right), \]  

(C8)

where \( I_{\cdot} \) is an indicator function and \( \hat{X}_1 \) follows one of the four types of generalized lognormal distributions. We first consider that \( \hat{X}_1 \) follows a regular lognormal distribution.

Assume that \( \hat{X}_1 \) has a regular lognormal distribution, namely \( \ln(\hat{X}_1) \sim N(\hat{\alpha}_1, \hat{\beta}_1^2) \), where \( \hat{\alpha}_1 \) and \( \hat{\beta}_1^2 \) can be computed via (25) in terms of \( E^{QC}[Y_1] \) and \( E^{QC}[Y_1^2] \) as follows:

\[ \hat{\alpha}_1 = \ln \left( E^{QC}[Y_1] \right) - \frac{1}{2} \hat{\beta}_1^2, \]
\[ \hat{\beta}_1^2 = \ln \left( E^{QC}[Y_1^2] \right) - 2 \ln \left( E^{QC}[Y_1] \right). \]

The second expectation of (C8) is computed as follows:

\[ E^{QC}[I_{\{\hat{X}_1 \geq K_1\}}] = P(\ln(\hat{X}_1) \geq \ln(K_1)) \]
\[ = N \left( \frac{\ln \left( E^{QC}[Y_1]/K_1 \right) - \frac{1}{2} \hat{\beta}_1^2}{\hat{\beta}_1} \right). \]  

(C9)
The first expectation of (C8) is computed as follows:

\[
E^{Q^C} \left[ \hat{X}_1 I_{\{\hat{X}_1 \geq K_1\}} \right] = E^{Q^C} \left[ e^{\ln(\hat{X}_1)} I_{[\ln(\hat{X}_1) \geq \ln(K_1)]} \right] \\
= E^{Q^C} \left[ e^{\hat{\beta}_1 Z + \hat{\alpha}_1} I_{[\hat{\beta}_1 Z + \hat{\alpha}_1 \geq \ln(K_1)]} \right] \quad \text{(where } Z \sim N(0, 1)) \\
= e^{\hat{\alpha}_1} \int_{\ln(K_1)/\hat{\beta}_1}^{\infty} e^{\hat{\beta}_1 z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz \\
= e^{\hat{\alpha}_1 + \frac{1}{2} \hat{\beta}_1^2} \int_{\ln(K_1)/\hat{\beta}_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(z-\hat{\beta}_1)^2} dz \quad \text{(let } U = Z - \hat{\beta}_1) \\
= e^{\hat{\alpha}_1 + \frac{1}{2} \hat{\beta}_1^2} N \left( \frac{\hat{\alpha}_1 + \hat{\beta}_1^2 - \ln(K_1)}{\hat{\beta}_1} \right) \\
= E^{Q^C} [Y_1] N \left( \frac{\ln \left( E^{Q^C} [Y_1] / K_1 \right) + \frac{1}{2} \hat{\beta}_1^2}{\hat{\beta}_1} \right). \quad \text{(C10)}
\]

Equations (C8)–(C10) lead to (43).

Second, consider a shifted lognormal distribution \( \hat{X}_1 = V + \hat{\gamma}_1 \), where \( \hat{\gamma}_1 \) is a constant and \( V \) is a regular lognormal distribution with \( \ln(V) \sim N(\hat{\alpha}_1, \hat{\beta}_1^2) \).
Equation (C7) can be rewritten as follows:

\[
\text{CPSO}_1(0) = V_d C(0; T_1, T_n) E^{Q^C} \left[ (\hat{X}_1 - K_1)^+ \right] \\
= V_d C(0; T_1, T_n) E^{Q^C} \left[ (V - (K_1 - \hat{\gamma}_1))^+ \right]. \quad \text{(C11)}
\]

By replacing \( K_1 \) in both (C9) and (C10) by \( K_1 + \hat{\gamma}_1 \), the CPSO1 pricing formula can be derived and is given in (44).

Third, let \( \hat{X}_1 \) be a negative lognormal distribution, namely \( \hat{X}_1 = -V \), where \( V \) is a regular lognormal distribution. Equation (C7) can be rewritten as follows:

\[
\text{CPSO}_1(0) = V_d \delta C(0; T_1, T_n) E^{Q^C} \left[ (-V - K_1)^+ \right], \quad \text{(C12)}
\]

\[
= V_d \delta C(0; T_1, T_n) \left( - E^{Q^C} [V I_{[V \leq -K_1]}] - K_1 E^{Q^C} [I_{[V \leq -K_1]}] \right). \quad \text{(C13)}
\]
Equation (C13) can be computed similarly to the first and second expectation in (C8), which leads to (45).

Fourth, let $\hat{X}_1$ be a negative-shifted lognormal distribution, namely $\hat{X}_1 = -(V + \hat{\gamma}_1)$, where $V$ is a regular lognormal distribution and $\hat{\gamma}_1$ is a constant. Equation (C7) can be rewritten as follows:

$$\text{CPSO}_1(0) = V_d C(0; T_1, T_n) E^{Q_T} [(-V - (K_1 + \hat{\gamma}))^+] ,$$  \hspace{1cm} (C14)

By replacing $K_1$ in (C12) by $K_1 + \hat{\gamma}_1$, the CPSO$_1$ pricing formula can be derived and is given in (46).

The pricing formulas of a CPSO$_2$ can be derived similarly and the derivation is omitted for the sake of brevity. We only provide the formulas for $E^{Q_C} [Y_2]$, $E^{Q_C} [Y_2^2]$, and $E^{Q_C} [Y_3]$, as follows:

$$E^{Q_C} [Y_2] = S_g(0, T_0) - S_f(0, T_0),$$  \hspace{1cm} (C15)

$$E^{Q_C} [Y_2^2] = S_g(0, T_0)^2 \exp \left( \int_0^{T_0} \| \tilde{\sigma}_g^0(t) \|^2 dt \right) + S_f(0, T_0)^2 \exp \left( \int_0^{T_0} \| \tilde{\sigma}_f^0(t) \|^2 dt \right) - 2 S_g(0, T_0) S_f(0, T_0) \exp \left( \int_0^{T_0} \tilde{\sigma}_g^0(t) \cdot \tilde{\sigma}_f^0(t) dt \right).$$  \hspace{1cm} (C16)

$$E^{Q_C} [Y_3^2] = S_g(0, T_0)^3 \exp \left( 3 \int_0^{T_0} \| \tilde{\sigma}_g^0(t) \|^2 dt \right) - S_f(0, T_0)^3 \exp \left( 3 \int_0^{T_0} \| \tilde{\sigma}_f^0(t) \|^2 dt \right) - 3 S_g(0, T_0)^2 S_f(0, T_0) \exp \left( \int_0^{T_0} \| \tilde{\sigma}_g^0(t) \|^2 + 2 \tilde{\sigma}_g^0(t) \cdot \tilde{\sigma}_f^0(t) dt \right) + 3 S_g(0, T_0) S_f(0, T_0)^2 \exp \left( \int_0^{T_0} \| \tilde{\sigma}_f^0(t) \|^2 + 2 \tilde{\sigma}_f^0(t) \cdot \tilde{\sigma}_g^0(t) dt \right).$$  \hspace{1cm} (C17)

APPENDIX D: EXAMINATION OF THE ACCURACY OF THE TWO APPROXIMATIONS

Because the resulting pricing formulas are derived based on two approximations for the LIBOR rate and the swap rate dynamics, this appendix intends to examine their accuracy. These two approximations are extensively used in practice and can also be found in Brace et al. (1997), Schlögl (2002), Wu and
Chen (2008, 2009b), and Brigo and Mercurio (2006). To conserve space in presentation, we only examine the domestic LIBOR rates and swap rates, and the results of the foreign rates are similar.

In Proposition 1, the domestic LIBOR rate under $Q$ is derived and specified by the following process:

\[
\frac{dL_d(t, T)}{L_d(t, T)} = \gamma_d(t, T) \cdot \sigma_{Bd}(t, T + \delta) dt + \gamma_d(t, T) \cdot d\tilde{W}(t),
\]  

(D1)

where $\sigma_{Bd}(t, T + \delta)$ is defined in (7). To make (D1) solvable, we approximate (D1) by the following process:

\[
\frac{dL_d(t, T)}{L_d(t, T)} = \gamma_d(t, T) \cdot \bar{\sigma}_{Bd}(t, T + \delta) dt + \gamma_d(t, T) \cdot d\tilde{W}(t),
\]

(D2)

where $\bar{\sigma}_{Bd}(t, T + \delta)$ is defined in (8).

To examine the accuracy of the approximation, we employ Monte Carlo (MC) simulation to simulate 30,000 samples of the LIBOR rates for each of two cases: the first case is implemented according to (D1), denoted by $L^{(i)}_d(T, T)$, and the second is based on (D2), denoted by $L^{(i)}_{d2}(T, T)$, where the index $i$ stands for the $i$th sample. The simulated LIBOR rates have ten different times to maturity, which are generated under eight different market scenarios, namely, using the market data on March 1, 2007, June 1, 2007, September 3, 2007, December 3, 2007, March 3, 2008, June 2, 2008, September 1, 2008, and December 1, 2008.\(^{17}\)

We compute the root-mean-square relative error (RMSRE) by

\[
\text{RMSRE} = \sqrt{\frac{\sum_{i=1}^{n} e_i^2}{n}},
\]

where $e_i = (L^{(i)}_{d1}(T, T) - L^{(i)}_{d2}(T, T))/L^{(i)}_{d1}(T, T)$, and the result is presented in Table III. Based on the simulation result, the approximation technique are accurate and robust over the past two years. Even on the most volatile date of the LIBOR rates, December 1, 2008, the largest percentage relative error of $L(10, 10)$ associated with the longer maturity date 10 is about 4%, which should still be acceptable in practice.

\(^{17}\)The market data can be obtained upon request from the authors, and the calibration procedure can refer to Wu and Chen (2011).
TABLE III
The Root-Mean-Square Relative Error Between Actual and Approximate LIBOR Rates

<table>
<thead>
<tr>
<th></th>
<th>March 1, 2007 (%)</th>
<th>June 1, 2007 (%)</th>
<th>September 3, 2007 (%)</th>
<th>December 3, 2007 (%)</th>
<th>March 3, 2008 (%)</th>
<th>June 2, 2008 (%)</th>
<th>September 1, 2008 (%)</th>
<th>December 1, 2008 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSRE</td>
<td>0.011</td>
<td>0.012</td>
<td>0.033</td>
<td>0.052</td>
<td>0.092</td>
<td>0.133</td>
<td>0.849</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.027</td>
<td>0.029</td>
<td>0.071</td>
<td>0.112</td>
<td>0.242</td>
<td>0.195</td>
<td>0.294</td>
<td>1.390</td>
</tr>
<tr>
<td>3</td>
<td>0.055</td>
<td>0.058</td>
<td>0.124</td>
<td>0.194</td>
<td>0.408</td>
<td>0.323</td>
<td>0.484</td>
<td>1.892</td>
</tr>
<tr>
<td>4</td>
<td>0.096</td>
<td>0.101</td>
<td>0.195</td>
<td>0.298</td>
<td>0.602</td>
<td>0.474</td>
<td>0.696</td>
<td>2.319</td>
</tr>
<tr>
<td>5</td>
<td>0.153</td>
<td>0.161</td>
<td>0.285</td>
<td>0.426</td>
<td>0.826</td>
<td>0.652</td>
<td>0.935</td>
<td>2.712</td>
</tr>
<tr>
<td>6</td>
<td>0.230</td>
<td>0.241</td>
<td>0.399</td>
<td>0.582</td>
<td>1.084</td>
<td>0.863</td>
<td>1.210</td>
<td>3.110</td>
</tr>
<tr>
<td>7</td>
<td>0.327</td>
<td>0.343</td>
<td>0.534</td>
<td>0.760</td>
<td>1.364</td>
<td>1.095</td>
<td>1.503</td>
<td>3.453</td>
</tr>
<tr>
<td>8</td>
<td>0.449</td>
<td>0.469</td>
<td>0.693</td>
<td>0.965</td>
<td>1.676</td>
<td>1.353</td>
<td>1.824</td>
<td>3.771</td>
</tr>
<tr>
<td>9</td>
<td>0.593</td>
<td>0.619</td>
<td>0.876</td>
<td>1.194</td>
<td>2.017</td>
<td>1.647</td>
<td>2.174</td>
<td>4.001</td>
</tr>
<tr>
<td>10</td>
<td>0.767</td>
<td>0.802</td>
<td>1.088</td>
<td>1.451</td>
<td>2.386</td>
<td>1.973</td>
<td>2.551</td>
<td>4.013</td>
</tr>
</tbody>
</table>

The accuracy of the LIBOR rate approximation is examined in this table. The simulated LIBOR rates have ten different times to maturity, and they are generated under eight different market scenarios, namely using the market data on March 1, 2007, June 1, 2007, September 3, 2007, December 3, 2007, March 3, 2008, June 2, 2008, September 1, 2008, and December 1, 2008. Each result is based on 30,000 sample paths.

Next, we test the accuracy of the swap rate approximation. We have shown that the swap rate dynamics follows the following process:

\[
\frac{dS_d(t, T_0)}{S_d(t, T_0)} = \sigma_{S_d}(t, T_0) \cdot d\tilde{W}(t), \tag{D3}
\]

where \(\sigma_{S_d}(t, T_0)\), defined in (A1). To make \(S_d(T_0, T_0)\) computable, we approximate (D3) by the following process:

\[
\frac{dS_d(t, T_0)}{S_d(t, T_0)} = \tilde{\sigma}_{S_d}^0(t, T_0) \cdot d\tilde{W}(t), \tag{D4}
\]

where \(\tilde{\sigma}_{S_d}^0(t, T_0)\) is defined in (18).

To examine the accuracy of the approximation, we employ MC simulation to simulate 30,000 samples of the five-year swap rates for each of the two cases: the first case is implemented according to (D3), denoted by \(S_{d1}^{(i)}(T, T)\), and the second is based on (D4), denoted by \(S_{d2}^{(i)}(T, T)\), where the index \(i\) stands for the \(i\)th sample and \(T = 5\). The simulated five-year swap rates are generated by using the same market data as in Table III.
The Root-Mean-Square Relative Error Between Actual and Approximate Swap Rates

<table>
<thead>
<tr>
<th>March 1, 2007 (%)</th>
<th>June 1, 2007 (%)</th>
<th>September 3, 2007 (%)</th>
<th>December 3, 2007 (%)</th>
<th>March 3, 2008 (%)</th>
<th>June 2, 2008 (%)</th>
<th>September 1, 2008 (%)</th>
<th>December 1, 2008 (%)</th>
</tr>
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<td>0.462</td>
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</table>

The accuracy of the swap rate approximation is examined in this table. The simulated five-year swap rates are generated under eight different market scenarios, namely using the market data on March 1, 2007, June 1, 2007, September 3, 2007, December 3, 2007, March 3, 2008, June 2, 2008, September 1, 2008, and December 1, 2008. Each result is based on 30,000 sample paths.

We compute the RMSRE by

$$\text{RMSRE} = \sqrt{\frac{\sum_{i=1}^{n} \epsilon_i^2}{n}},$$

where \( \epsilon_i = (S_{d1}^{(i)}(T, T) - S_{d2}^{(i)}(T, T))/S_{d1}^{(i)}(T, T) \), and the result is presented in Table IV. According to the simulation result, the approximation technique are accurate and robust over the past two years. In addition, because the swap rates are the weighted averages of LIBOR rates, the average effect makes the swap rate approximation more stable among different lengths of times to maturity.

**BIBLIOGRAPHY**


